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A Hopf laboratory for symmetric functions

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Abstract

An analysis of symmetric function theory is given from the perspective of the underlying Hopf and bi-algebraic structures. These are presented explicitly in terms of standard symmetric function operations. Particular attention is focused on Laplace pairing, Sweedler cohomology for 1- and 2-cochains and twisted products (Rota cliffordizations) induced by branching operators in the symmetric function context. The latter are shown to include the algebras of symmetric functions of orthogonal and symplectic type. A commentary on related issues in the combinatorial approach to quantum field theory is given.

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1. Introduction

The symmetric group, and its role in representation theory and the related symmetric polynomials, is central to many descriptions of physical phenomena, from classical through to statistical and quantum domains. The present work is an initial attempt to synthesize aspects of symmetric function theory from the view point of the structure theory of underlying Hopf algebras and bialgebras. That such a deeper framework is available is well recognized (for references see below). However, our aim is to exploit the Hopf algebra theory as fully as possible. Specifically, our interest is in deformed products and coproducts, and their characterization by cohomological techniques in the context of symmetric functions.

There are several motivations for an approach of this nature. In the first place, the symmetric functions provide a concrete arena and convenient laboratory for the structures of interest suggested by Hopf algebras. Furthermore, the Hopf versions of symmetric function interrelationships confer systematic explanatory insights, and considerable scope for generalizations. Last but not least, it is our intention to provide specific material to an audience of mathematical physicists and other practitioners, for whom it is valuable to meld concrete constructs with abstract developments, which we feel are of abiding importance in several areas of application.

Our own primary motivation for the present study is that symmetric functions may well serve as a simplified model for combinatorial approaches to quantum field theory (QFT). The triply iterated structure seen in QFT is perfectly mirrored in symmetric functions. Questions such as time versus normal ordering, and renormalization, can be framed in abstract combinatorial and algebraic terms (for references see below). The precise analogues in symmetric functions should be concrete and calculable, and may serve as a germ to understand the more complex QFT setting. This work provides the groundwork for such a study.

The outline of this paper is as follows. An analysis of symmetric function theory is developed, from the perspective of the underlying Hopf and bi-algebraic structures. These are presented explicitly in terms of standard symmetric function notation (section 2). The implications of Laplace pairings for symmetric function operations and expansions are detailed, and exemplified for the case of Kostka matrices (section 2). In section 3, Sweedler's cohomology (for references see below) is discussed, with emphasis on the analysis of 1- and 2-cochains, cocycle conditions and coboundaries, and these are shown to control deformed products via Rota cliffordization (section 4). Associative cliffordizations (derived from 2-cocycles, modulo a 2-coboundary) include cases isomorphic to standard multiplication (but non-isomorphic as augmented algebras), and also the Newell–Littlewood product for symmetric functions of orthogonal and symplectic type. The relation between the latter and the standard outer symmetric function algebra is established, using certain classes of branching operators associated with symmetric function infinite series, whose properties are discussed. These include some of the so-called 'remarkable identities' known for these series from applications to representation theory. Finally, an appendix on the elements of λ-rings for the explicit case of symmetric functions is included. The paper concludes with a discussion of the main results, the outlook for the approach and further elaboration of the links between the above constructs and the combinatorial approach to QFT.

2. The Hopf algebra of symmetric functions

2.1. Background and notation

As mentioned above, we wish to develop aspects of the theory of symmetric functions wherein the underlying Hopf structures, or at least bialgebraic structures, are exploited. It is well known that the symmetric functions form a Hopf algebra [40, 41, 44, 47]. However, much of the literature uses abstract λ-ring notation (see the appendix) and is hence not easily appreciated by a practitioner. The abstract approach presented in [36, 39] is designed for the framework of quasi-symmetric functions and the permutation group. Although some of the present Hopf and bi-algebras are factor algebras, the technical complications of this much more general setting obscure the explicit approach to symmetric functions which we espouse.

In particular, our interest focuses on *cliffordization* and other *deformed* products and coproducts. Cliffordization was introduced by Gian-Carlo Rota and Joel Stein [40], but is related to a Drinfeld twist [9] in the sense of Sweedler [46]. From Sweedler's approach, one learns that many properties of a deformed product can be characterized by cohomological methods. We are going to employ this for symmetric functions. The paper of Rota and Stein, *loc cit*, has as its main motivation the introduction of *plethystic algebras* in a very general setting. However, in the language of letter-place superalgebras, it is again difficult to access concrete calculations in terms appreciated by practitioners. The approach by Thibon [47] and Scharf and Thibon [44] does use the Hopf structure, however, without exploiting Hopf algebra theory (in fact making only perfunctory use of the additional coalgebra structures). For reasons of accessibility, and for a reasonably self-contained presentation, we therefore introduce Hopf- and bi-algebras for symmetric functions in a self-consistent manner here. For a QFT-based account of combinatorial issues entering into generating functional formulations, see [9, 10]. At several points in what follows, relevant analogies will be pointed out (see also the concluding remarks).

We use mainly the notation and definitions of Macdonald [35]. The $\mathbb{Z}$-graded space of symmetric functions is denoted as $\Lambda = \oplus_n \Lambda^n$ where the Λ^n are those subspaces having n variables $(x_1, x_2, \ldots, x_n)$. A partition of an integer is given either as a non-increasing list of its parts with possible trailing zeros $\lambda = (\lambda_1, \ldots, \lambda_s)$, using round parentheses, or may be given as $\lambda = [r_1, \ldots, r_n] = 1^{r_1}2^{r_2}\ldots n^{r_n}$ where the r_i count the occurrences of parts i in λ. The transposed or conjugated partition λ' is obtained by mirroring the Ferrer's diagram at the main diagonal. The length of a partition $\ell(\lambda)$ is the number of its (nonzero) parts, and the weight $|\lambda|$ or ω_λ is given by the sum of its parts, $\lambda_1 + \lambda_2 + \cdots + \lambda_s = |\lambda| = 1r_1 + 2r_2 + \cdots + nr_n$. A further way to describe partitions is given by Frobenius notation. This gives the length of 'arms' α_i and 'legs' β_i of a diagram of a partition measured from the main diagonal. The length—number of boxes—of the main diagonal is the Frobenius rank r of a partition, $\lambda = (\alpha_1, \ldots, \alpha_r \mid \beta_1, \ldots, \beta_r)$. Thus, for example $\lambda = (5, 4, 2, 2, 2, 1) = [1, 3, 0, 1, 1, 0, \ldots] = (4, 2 \mid 5, 3)$.

It is convenient to introduce various bases in the ring Λ. The complete symmetric functions will be denoted as $h_\lambda = h_{\lambda_1}\ldots h_{\lambda_s}$ with generating function $H(t) = \sum_n h_n t^n = \prod_{i=1}^{\infty}(1 - x_i t)^{-1}$. The elementary symmetric functions are defined as $e_\lambda = e_{\lambda_1}\ldots e_{\lambda_s}$ with generating function $E(t) = \sum_n e_n t^n = \prod_{i=1}^{\infty}(1 + x_i t)$. Note that the h_r are represented by a diagram with a single row, and the e_r as a single column. For the definition of further Schur function series, see below. We need furthermore the monomial symmetric functions $m_\alpha = \sum_{w\in S_n} x_{w(1)}^{\alpha_1}\ldots x_{w(r)}^{\alpha_r}$, and the power sum symmetric functions $p_\lambda = p_{\lambda_1}\ldots p_{\lambda_r}$, where $p_n = \sum x_i^n$. Note that the p_λ form a $\mathbb{Q}$-basis only, but see [22]. The most important basis for applications is that of Schur or S-functions, denoted as s_λ. Schur functions (also homogenous symmetric polynomials) can be defined via the Jacobi–Trudi determinantal formulae from the complete or elementary symmetric functions, or as a ratio of a determinant of monomials with the van der Monde determinant (see [35]). Occasionally it is convenient to adopt the Littlewood notation $\{\lambda\}$ for the Schur function s_λ.

2.2. Addition, products and plethysm

Reference to the 'ring' of symmetric functions amounts to saying that there are two mutually compatible binary operations. The *addition* is the conventional addition of polynomial functions and the *multiplication*, the conventional product of polynomial functions, is the so-called outer product of symmetric functions. In terms of Schur functions it is described by the well-known Littlewood–Richardson rule on the diagrams of the factors:

$$\mathsf{M}(s_\lambda \otimes s_\mu) = s_\lambda \cdot s_\mu = \sum_\nu C^\nu_{\lambda\mu} s_\nu \qquad s_\lambda \cdot s_\mu = s_\mu \cdot s_\lambda \tag{2.1}$$

wherein the dot will sometimes be omitted; the capital M is retained to denote the outer product map. The *unit* for this product 1_{M} is the constant Schur function, corresponding to the empty or null partition $s_0 = 1$, sometimes just denoted as 1 in what follows. The Littlewood–Richardson coefficients $C^\nu_{\lambda\mu} = C^\nu_{\mu\lambda}$ may be addressed as a multiplication table, with non-negative integer coefficients since they count the number of lattice paths from ν to λ under some restrictions imposed by μ or vice versa. The coefficient is zero unless $|\nu| = |\lambda| + |\mu|$.

There is another product on symmetric functions, the *inner product*, denoted by lower case m or $\star$, which is displayed most conveniently in the power sum basis

$$\mathsf{m}(p_\lambda \otimes p_\mu) = p_\lambda \star p_\mu = \delta_{\lambda\mu} z_\lambda p_\lambda \tag{2.2}$$

where $z_\lambda = \prod i^{r_i} r_i!$ with $\lambda = [r_1, \ldots, r_n]$ (and $z_{(n)} \equiv z_n = n$). The unit for the inner product can be given in the power sum basis as

$$1_{\mathsf{m}} := \sum_n \frac{p_n}{z_n} \qquad \text{since} \qquad 1_{\mathsf{m}} \star p_i := \sum_n \frac{p_n}{z_n} \star p_i = \sum_n \frac{\delta_{ni} z_i}{z_n} p_i = p_i. \tag{2.3}$$

Alternatively, the unit reads in the Schur function basis

$$1_{\mathsf{m}} = \sum_{n\geqslant 0} s_{(n)} = \sum_{n\geqslant 0} h_n. \tag{2.4}$$

A variation on the outer multiplication is the notion of symmetric function (right or left) *skew* defined for Schur functions by

$$s_{\lambda/\mu} = \sum_\nu s_\nu C^\lambda_{\nu\mu} \qquad s_{\mu\backslash\lambda} = \sum_\nu C^\lambda_{\mu\nu} s_\nu \qquad s_{\lambda/\mu} = s_{\mu\backslash\lambda} \tag{2.5}$$

for which necessarily $|\lambda| \geqslant |\mu|$ (there is no corresponding *inner* skew because the corresponding structure coefficients for the inner multiplication are totally symmetrical).

Besides addition and the two products, one can define composition or *outer plethysm*, denoted as $\circ$, on symmetric functions. This reads in terms of power sums as

$$g \circ p_n = g\big(x_1^n, x_2^n, \ldots\big) \quad (= p_n \circ g) \qquad p_n \circ p_m = p_{nm}. \tag{2.6}$$

Finally, for the definition of *inner plethysm* in a Hopf algebra approach see Scharf and Thibon [44]. All connectivities will play a joint role in what follows.

2.3. Schur scalar product

It is convenient to introduce a scalar product on symmetric functions. The prominent role which the Schur functions play is reflected in that they form an orthonormal basis with respect to the scalar product $(\cdot \mid \cdot)$, by definition

$$(s_\lambda \mid s_\mu) = \delta_{\lambda\mu}. \tag{2.7}$$

Furthermore, the p_λ form an orthogonal basis only,

$$(p_\lambda \mid p_\mu) = z_\lambda \delta_{\lambda\mu} \tag{2.8}$$

with z_λ as in (2.2). The scalar product allows us to define dual elements in a unique way[3]. Hence, any symmetric function f can be expanded into a basis, for example, in the Schur function basis or the power sum basis

$$f = \sum_\lambda (s_\lambda \mid f) s_\lambda = \sum_\lambda \frac{(p_\lambda \mid f)}{z_\lambda} p_\lambda. \tag{2.9}$$

The scalar product can be used to define adjoints. If F is an operator on the space of symmetric functions, then we define $(s_\lambda \mid F(s_\mu)) = (G(s_\lambda) \mid s_\mu)$. In operator theory G would be denoted as F^*, but we will have occasion to use several generic maps where the adjoints have their own names.

[3] In any finite collection $\oplus_{\mathsf{finite}} \Lambda^n$ which has to be dense in the limit $n \to \infty$.

2.4. Variables versus tensor products

We have generally omitted the explicit variables in functions such as s_λ. However, we may reconsider this habit as follows. We have until now used *only one* species of variables, namely the linearly ordered set $\{x_i\}$. We may collect these into a (possibly infinite) set or formal variable X. All of the above statements having no variable may then be re-read as '...with all variables from the set X'. However, it turns out that all notions make perfect sense if the symmetric functions are considered as *operators* on the formal variable X, see [35] chapter 1 appendix. This is the so-called λ-ring notation (see the appendix)[4]. However, one should note the important change in the realm of the statements made in this language. We are now ready to introduce a second set of variables Y disjoint from X, and we can consider symmetric functions on formal sums $X + Y$ or formal products XY, namely the sets $\{x_i, y_j\}$ or $\{x_i y_j\}$. It is well known that one can give an isomorphism θ between such multi-variable settings, and tensor products on $\mathsf{End}\,\Lambda \cong \Lambda \otimes \Lambda$ since $\Lambda \cong \Lambda^*$,

$$\theta : F(X)G(Y) \to F(X) \otimes G(X) \in \Lambda \otimes \Lambda. \tag{2.10}$$

In other words, the X and Y keep track of the tensor slot in a tensor product. This is the origin of the letter-place idea promoted in [22]. For our purpose, it is enough to use this identification to and fro for convenience, and to make contact with the literature.

As a further step, we consider tensor products of Λ forming the tensor algebra $\mathsf{Tens}[\Lambda] \equiv \Lambda^{\otimes} = \oplus_n \Lambda^{\otimes^n}$. Due to the identification made by the Schur scalar product, we find that endormorphisms of symmetric functions are elements of $\Lambda \otimes \Lambda$, with the second factor seen as dual. The endomorphic product is then composition

$$\begin{aligned} &\circ : (\Lambda \otimes \Lambda) \otimes (\Lambda \otimes \Lambda) \to (\Lambda \otimes \Lambda) \\ &(G \circ H)(X) = G(H(X)). \end{aligned} \tag{2.11}$$

In terms of letter-place algebras, Λ is generated by a single alphabet[5] X, while $\Lambda^{\otimes}$ is generated by an infinite collection of disjoint alphabets, hence $\Lambda^{\otimes} \cong \mathsf{Tens}\ \mathsf{Tens}[x_1 + x_2 + x_3 + \cdots]$. In this way, using associativity, we can extend the various structures obtained in Λ to $\Lambda^{\otimes}$. More technically speaking, $\Lambda^{\otimes}$ provides a symmetric monoidal category and the product and coproduct maps M, m, Δ, δ (see below) are morphisms on $\Lambda^{\otimes}$.

2.5. Inner and outer coproducts

The canonical extension of the Schur scalar product to tensor powers of Λ is

$$\begin{aligned} &(.\mid.) : \Lambda^{\otimes} \otimes \Lambda^{\otimes} \to \mathbb{Z} \\ &(.\mid.)\,|_{\Lambda^{\otimes^r} \otimes \Lambda^{\otimes^s}} = \delta_{rs} \prod_k (.\mid.)_k \end{aligned} \tag{2.12}$$

where $(.\mid.)_k$ denotes the scalar product in $\Lambda \otimes \Lambda$ applied to the kth factors on each side.

We use this scalar product to dualize the outer and inner products, and so define the outer coproduct Δ and the inner coproduct δ—once more distinguished by case (notation from [44]):

Definition 2.1.

$$\begin{aligned} &(\Delta F \mid G \otimes H) = (F \mid GH) \\ &(\delta F \mid G \otimes H) = (F \mid G \star H) \end{aligned} \tag{2.13}$$

[4] The name may originate from λ-calculus, where one has a 'for all' quantifier establishing exactly the meaning given in this section.

[5] Possibly of only a single letter!

Specifically, inserting a power sum basis, we can read that

$$\begin{aligned} \Delta p_i &= p_i \otimes 1 + 1 \otimes p_i = \theta(p_i(X+Y)) \\ \delta p_i &= p_i \otimes p_i = \theta(p_i(XY)) \end{aligned} \tag{2.14}$$

and hence infer that these properties lift to Schur functions s_λ or generic symmetric functions f, g. Since they are dualized from associative products, these coproducts are coassociative, and we can define the iterated coproducts

$$\Delta^0 = \mathrm{Id} \qquad \Delta^1 = \Delta \qquad \Delta^r = (\Delta \otimes \mathrm{Id}) \circ \Delta^{r-1} = (\mathrm{Id} \otimes \Delta) \circ \Delta^{r-1} \tag{2.15}$$

and analogously for the inner coproduct δ^r. The outer coproduct Δ reads on Schur functions in particular

$$\Delta(s_\lambda) = \sum_\alpha s_{\lambda/\alpha} \otimes s_\alpha = \sum_{\alpha\beta} C^\lambda_{\alpha\beta} s_\beta \otimes s_\alpha \tag{2.16}$$

where α, β run over all possible partitions (however, only a finite number of terms contribute). Note, that the Littlewood–Richardson coefficients now make up the *comultiplication table*[6].

It is very convenient to hide the complexity of indexing of coproducts away via Sweedler notation,

$$\begin{aligned} \Delta(a) &= \sum_{(a)} a_{(1)} \otimes a_{(2)} \\ (\Delta \otimes \mathrm{Id}) \circ \Delta(a) &= \sum_{(a)} a_{(1)} \otimes a_{(2)} \otimes a_{(3)} \end{aligned} \tag{2.17}$$

where the sum is also usually suppressed. If a distinction between Sweedler indices is needed, we may use the Brouder–Schmitt convention [10] that

$$\Delta(a) = \sum_{(a)} a_{(1)} \otimes a_{(2)} \qquad \delta(a) = \sum_{(a)} a_{[1]} \otimes a_{[2]} \tag{2.18}$$

keeping track of the type of coproduct involved. If partitions are involved, as for example in Schur functions, we write simply

$$\Delta(s_\lambda) = s_{\lambda(1)} \otimes s_{\lambda(2)} \qquad \delta(s_\lambda) = s_{\lambda[1]} \otimes s_{\lambda[2]}. \tag{2.19}$$

Having the scalar product and coproducts in hand, a natural status for the symmetric function *skew* product can be recognized[7], as the natural action of *dual* elements of Λ derived from the outer coproduct:

$$\begin{aligned} s_{\mu\backslash\lambda} &= ((\mu|.) \otimes \mathrm{Id}) \circ \Delta(\lambda) = (s_{\lambda(1)} \mid \mu) s_{\lambda(2)} \\ &= s_{\lambda(1)}(\mu \mid s_{\lambda(2)}) = (\mathrm{Id} \otimes (\mu|.)) \circ \Delta(\lambda) = s_{\lambda/\mu}. \end{aligned} \tag{2.20}$$

As noted already, for the *inner* coproduct this dual action is identical to $\star$ itself.

Definition 2.2. *The counits $\epsilon^\Delta, \epsilon^\delta$ of the two coproducts are*

$$\begin{aligned} &\epsilon^\Delta(p_\lambda) := \delta_{\lambda,0} \\ &(\epsilon^\Delta \otimes \mathrm{Id}) \circ \Delta(p_n) = \epsilon^\Delta(p_n) \otimes 1 + \epsilon^\Delta(1) \otimes p_n = p_n \end{aligned} \tag{2.21}$$

[6] They should be called *section* coefficients in this context, and the indices should be arranged as $C^{\alpha\beta}_\lambda$. We stay, however, with the standard convention to prevent possible confusion.

[7] Technically, the adjoint of outer product as an element of $\mathsf{End}(\Lambda)$—the so-called Foulkes derivative (see Macdonald *loc cit*).

$$\begin{aligned} &\epsilon^{\delta}(p_{\lambda}) := 1 \\ &(\epsilon^{\delta} \otimes \mathrm{Id}) \circ \delta(p_n) = \epsilon^{\delta}(p_n) \otimes p_n = p_n. \end{aligned} \tag{2.22}$$

In summary, the unit of the outer product is the constant symmetric function s_0, and the corresponding outer counit is the projection onto s_0; the unit of the inner product is given by the series $H(t)$ at $t = 1$, while the inner counit is given by projecting all power sums to 1.

2.6. Hopf algebra and bialgebra structures

Since we deal with symmetric products and a self-dual space with respect to the Schur scalar product, we can verify that $\Lambda^{\otimes}$ is a symmetric tensor category with trivial braiding[8], i.e. for $V, W \in \Lambda^{\otimes}$

$$\mathsf{sw}(V \otimes W) = W \otimes V. \tag{2.23}$$

Given the two products, outer M and inner m, and the two coproducts, outer Δ and inner δ, it is natural to investigate which pairs have additional bialgebra or Hopf algebra structure. Moreover, it is well known from Hopf algebra theory that one can form convolution products from a pair of a coproduct and a product, so we have four possible convolutions and the question arises as to which of these convolutions admit antipodes.

Case 1. The outer product and outer coproduct M, Δ:

Theorem 2.3. *The septuple $H = (\Lambda, \mathsf{M}, 1_{\mathsf{M}}, \Delta, \epsilon^{\Delta}, \mathsf{sw}, \mathsf{S})$ is a Hopf algebra (denoted as the outer Hopf algebra of symmetric functions).*

Proof. We know already associativity, coassociativity and unit, counit from which the convolutive unit follows, so we need to show (i) the compatibility axiom for product and coproduct to form a bialgebra, (ii) the existence of the antipode.

(i) Consider the image of $\Delta(s_{\lambda})$ under θ^{-1},

$$s_{\lambda}(x, y) = \sum_{\alpha} s_{\alpha}(x) s_{\lambda/\alpha}(y) = \theta^{-1}(s_{\lambda(1)} \otimes s_{\lambda(2)}). \tag{2.24}$$

Computing the following product in two different ways gives

$$\begin{aligned} (a) \quad s_{\lambda}(x, y) s_{\mu}(x, y) &= s_{\lambda/\alpha}(x) s_{\alpha}(y) s_{\mu/\beta}(x) s_{\beta}(y) \\ &= s_{(\lambda/\alpha)\cdot(\mu/\beta)}(x) s_{\alpha\cdot\beta}(y) = s_{\lambda\cdot\mu/\alpha\cdot\beta}(x) s_{\alpha\cdot\beta}(y) \\ (b) \quad s_{\lambda}(x, y) s_{\mu}(x, y) &= s_{\lambda\cdot\mu}(x, y) \\ &= s_{\lambda\cdot\mu/\rho}(x) s_{\rho}(y) \\ &\Leftrightarrow s_{\lambda\cdot\mu(1)} \otimes s_{\lambda\cdot\mu(2)} = s_{\lambda(1)} s_{\mu(1)} \otimes s_{\lambda(2)} s_{\mu(2)}. \end{aligned} \tag{2.25}$$

From $(a) = (b)$ we can conclude that the product is a coalgebra homomorphism, and the coproduct is an algebra homomorphism, showing the compatibility axiom.

(ii) We have to show that the antipode S defined as

$$\sum_{\alpha} \mathsf{S}(s_{\alpha}) \cdot s_{\lambda/\alpha} = 1_{\mathsf{M}} \circ \epsilon^{\Delta}(s_{\lambda}) = \delta_{\lambda 0} \tag{2.26}$$

[8] Hall–Littlewood symmetric functions and q-Kostka–Foulkes polynomials would be associated with the introduction of a non-trivial grade group (see the concluding remarks below).

exists. This can be done by using a recursive argument as in Milnor and Moore [37]. From this one obtains in lowest orders that

$$\mathsf{S}(s_\lambda) = (-1)^{|\lambda|} s_{\lambda'}. \tag{2.27}$$

From generating functions we know that in Λ,

$$\begin{aligned} &H(t)E(-t) = 1 \\ &\sum_r (-1)^{n-r} h_r e_{n-r} = \delta_{n0} = \sum_r (-1)^{n-r} s_{(r)} s_{(1^{n-r})} \end{aligned} \tag{2.28}$$

which could be extended[9] to $\Lambda^\otimes$ via the Jacobi–Trudi formulae; however, we will take another route. Using λ-ring notation we find with Macdonald [35, pp 29–43] that in $\Lambda^\otimes$ the following holds:

$$\begin{aligned} s_\lambda(x, y) &= \sum_\mu s_{\lambda/\mu}(x) s_\mu(y), = \sum_{\mu,\nu} C^\lambda_{\mu\nu} s_\mu(x) s_\nu(y) \\ s_\lambda(X + Y) &= \sum_\mu s_{\lambda/\mu}(X) s_\mu(Y) \\ s_\lambda(X - X) &= \sum_\mu s_{\lambda/\mu}(X) s_\mu(-X) \\ s_\lambda(0) &= \sum_\mu (-1)^{|\mu|} s_{\lambda/\mu}(x) s_{\mu'}(x) \end{aligned} \tag{2.29}$$

from which we obtain the desired result[10] noting that $s_{(0)}(0) = 1$ and $s_\lambda(0) = 0$ for $\lambda \neq (0)$. □

Note that the antipode is related up to a sign factor to the ω-involution of Macdonald, which yields just the transpose of the partition, $\omega(s_\lambda) = s_{\lambda'}$. It is this sign factor which turns the antipode into a Möbius-like function, inherited from the underlying poset structure of the lattice of diagrams.

Finally, note that the coproduct Δ may look quite different in other bases, e.g., in the power sums we find as noted already

$$\begin{aligned} &\Delta(p_n) = p_n \otimes 1 + 1 \otimes p_n \qquad \Delta(1) = 1 \otimes 1 \\ &\Delta(p_n^2) = \Delta(p_n)\Delta(p_n) = p_n^2 \otimes 1 + 2p_n \otimes p_n + 1 \otimes p_n \\ &(\Delta \otimes \mathrm{Id})\Delta(p_n) = p_n \otimes 1 \otimes 1 + 1 \otimes p_n \otimes 1 + 1 \otimes 1 \otimes p_n. \end{aligned} \tag{2.30}$$

Generally

$$\begin{aligned} \Delta(p_n) &= \sum_{k=0}^{r} \frac{r!}{k!(r-k)!} p_n^k \otimes p_n^{r-k} = \sum_{k=0}^{r} \binom{r}{k} p_n^k \otimes p_n^{r-k} \\ \Delta^{(l-1)}(p_n) &= \sum_{\sum k_i = r} \frac{r!}{k_1! \ldots k_l!} p_n^{k_1} \otimes p_n^{k_2} \otimes \cdots \otimes p_n^{k_l}. \end{aligned} \tag{2.31}$$

[9] Here and subsequently certain steps are framed in $\Lambda^\otimes$ rather than Λ. The $^\otimes$ is occasionally omitted by abuse of notation.

[10] $s_\lambda(x, -x)_n$ for a finite number of variables can be regarded as a symmetric function with compound argument $(x_i y_j)$. Expanding w.r.t. $\{y_j\} = \{1, -1\}$ evaluates the superdimension of representations of $GL(1/1)$ occurring in a branching [12] from $GL(n/n)$ to $GL(n) \times GL(1/1)$. These are all two-dimensional and typical, unless $\lambda = 0$, from which the result follows.

Aside. Relations of this type are quite common in 'finite operator calculus' [42]. The above 'addition theorem' (see [35], p 43) is an analogue of the so-called Appell and Scheffer sequences.

Case II. The outer product and inner coproduct M, δ:

Theorem 2.4. *The algebra* $A = (\Lambda, \mathsf{M}, 1_{\mathsf{M}})$ *and the coalgebra* $C = (\Lambda, \delta, \epsilon^{\delta})$ *form a bialgebra, but not a Hopf algebra.*

Proof. We compute firstly the homomorphism property for the bialgebra,

$$\begin{aligned}
&(a) \quad \delta \circ \mathsf{M}(p_n \otimes p_m) = \delta(p_{nm}) = p_{nm} \otimes p_{nm} \\
&(b) \quad \mathsf{M} \otimes \mathsf{M}(\mathrm{Id} \otimes \mathsf{sw} \otimes \mathrm{Id})(\delta \otimes \delta)(p_n \otimes p_m) \\
&\qquad = \mathsf{M} \otimes \mathsf{M}(p_n \otimes p_m \otimes p_n \otimes p_m) \\
&\qquad = p_{nm} \otimes p_{nm}
\end{aligned} \tag{2.32}$$

which shows $(a) = (b)$ as needed. The antipode has to fulfil

$$\mathsf{S}(p_n)p_n = 1_{\mathsf{M}} \circ \epsilon^{\delta}(p_n) = \sum_m \frac{p_m}{z_m}. \tag{2.33}$$

Since the right-hand side contains terms of all grades m, but the left-hand side terms only have grades which are multiples of n, and since the power sums are independent, this requirement cannot be fulfilled. □

Case III. The inner product and outer coproduct m, Δ:

Theorem 2.5. *The algebra* $A = (\Lambda, \mathsf{m}, 1_{\mathsf{m}})$ *and the coalgebra* $C = (\Lambda, \Delta, \epsilon^{\Delta})$ *form a bialgebra, but not a Hopf algebra.*

Proof. We compute firstly the homomorphism property for the bialgebra,

$$\begin{aligned}
&(a) \quad \Delta \circ \mathsf{m}(p_n \otimes p_m) = \Delta(\delta_{nm} z_n p_n) = \delta_{nm} z_n (p_n \otimes 1 + 1 \otimes p_n) \\
&(b) \quad \mathsf{m} \otimes \mathsf{m}(1 \otimes \mathsf{sw} \otimes 1)(\Delta \otimes \Delta)(p_n \otimes p_m) \\
&\qquad = \mathsf{m} \otimes \mathsf{m}(p_n \otimes p_m \otimes 1 \otimes 1 + p_n \otimes 1 \otimes 1 \otimes p_m + 1 \otimes p_m \otimes p_n \otimes 1 \\
&\qquad\quad + 1 \otimes 1 \otimes p_n \otimes p_m) \\
&\qquad = z_n \delta_{nm} p_n \otimes 1 + z_n \delta_{nm} z_n 1 \otimes p_n
\end{aligned} \tag{2.34}$$

which shows $(a) = (b)$ as needed. The antipode $\mathsf{S}^{\Delta,\mathsf{m}}$ has to fulfil the following requirement

$$\mathsf{S}(p_n) \star 1 + \mathsf{S}(1) \star p_n = 1_{\mathsf{m}} \circ \epsilon^{\Delta}(p_n). \tag{2.35}$$

Firstly let $n = 0$, then the right-hand side reduces to 1_{m}, while the left-hand side is $2\mathsf{S}(1)1$, which implies that $2\mathsf{S}(1) = 1_{\mathsf{m}} = \sum_n p_n/z_n$. Hence, we find in the case $n \neq 0$

$$\mathsf{S}(p_n) \star 1 + \frac{1}{2} 1_{\mathsf{m}} \star p_n = \delta_{n0} \sum_m \frac{p_m}{z_m} \tag{2.36}$$

which cannot be fulfilled. □

Case IV. The inner product and inner coproduct m, δ:

Theorem 2.6. *The coalgebra* $C = (\Lambda, \delta, \epsilon^{\delta})$ *and the algebra* $A = (\Lambda, \star, 1_{\mathsf{m}})$ *do not form a Hopf algebra, and not even a bialgebra.*

Table 1. Mutual product–coproduct homomorphisms.

I	$(\Delta, \mathsf{M}, \mathsf{S})$	Outer Hopf algebra
II	(δ, M)	Bialgebra
III	(Δ, m)	Bialgebra
IV	(δ, m)	Inner convolution

Proof. To show it is not Hopf, it suffices to see that there does not exist an antipode. We use the power sum basis. Assuming the antipode is described by the operator $\mathsf{S}(p_n) = \sum_m \mathsf{S}_{nm} p_m$, we can compute

$$\begin{aligned}
&\mathsf{S}(p_n) \star p_n = 1_{\mathsf{m}} \circ \epsilon^{\delta}(p_n) \\
&\sum_m \mathsf{S}_{nm} p_m \star p_n = \sum_m \frac{1}{z_m} p_m \\
&z_n \mathsf{S}_{nn} p_n = \sum_m \frac{1}{z_m} p_m
\end{aligned} \tag{2.37}$$

which cannot be fulfilled due to the linear independence of the power sum functions. To show that the structure is not a bialgebra we have to show that this pair of product and coproduct is not mutually homomorphic, for example

$$\begin{aligned}
(a)\quad & \delta \circ \mathsf{m}(p_n \otimes p_m) = \delta\left(\frac{\delta_{nm}}{z_n} p_n\right) = \delta_{nm} z_n p_n \otimes p_n \\
(b)\quad & \mathsf{m} \otimes \mathsf{m}\,(\mathrm{Id} \otimes \mathsf{sw} \otimes \mathrm{Id})\,(\delta \otimes \delta)(p_n \otimes p_m) \\
& \qquad = \mathsf{m} \otimes \mathsf{m}(p_n \otimes p_m \otimes p_n \otimes p_m) = \delta_{nm} z_n^2 (p_n \otimes p_n).
\end{aligned} \tag{2.38}$$

One cannot fulfil $(a) = (b)$ due to the fact that the coproduct (or the product) would need to be rescaled with $\sqrt{z_n}$, which drops out of the ring of integers, or even the quotient field $\mathbb{Q}$ of rational numbers. □

Aside. The lack of having an antipode in this case may not be so surprising as at first sight. Remember that $\theta^{-1}(\delta(p_\lambda(X))) = p_\lambda(XY)$. Hence the antipode would require $Y = 1/X$, which is beyond the ring Λ of symmetric functions[11]. The present problem may hence be dubbed the 'localization problem' of the inner product, in analogy with the 'localization' process of enlarging a ring to its quotient field in algebraic geometry. Using divided powers may provide a cure (see [22]).

The cases I–IV are summarized in table 1. Evidently the presence of inner (co)products decreases the compatibility with Hopf algebra axioms. One can think about a slightly altered definition of an *inner antipode* which would cure this, and allow four mutually related Hopf algebras[12].

2.7. Scalar product—Laplace pairing

A pairing is in general a map $\pi : A \otimes B \to C$. Particular pairings are actions $\bullet : G \otimes M \to M$, multiplications $\mu : A \otimes A \to A$ or evaluations $\mathsf{ev} : A \otimes A \to \mathbb{Z}$. A pairing which is compatible with a coproduct, i.e. forming a bialgebra, is called after Sweedler a measuring (for a theory

[11] The introduction of X^{-1} as formal variables is considered, for example, in Crapo and Schmitt [11] or Borcherds [7].

[12] The candidates we have in mind are ζ-functions and Möbius functions, where the Möbius function replaces the antipode. This still fits into the theory of λ-rings [30].

of pairings and measurings see [45]). We will use the scalar-valued case in the present work only, but note some more general cases.

Definition 2.7. *A pairing is called a Laplace pairing [22] if it enjoys the following two properties:*

$$\begin{aligned} &(i)\ (w \mid a \cdot b) = (w_{(1)} \mid a)(w_{(2)} \mid b) \\ &(ii)\ (a \cdot b \mid w) = (a \mid w_{(1)})(b \mid w_{(2)}) \end{aligned} \tag{2.39}$$

and if the product and coproduct are mutually homomorphisms.

The name stems from the fact that these identities imply the expansion rules for determinants in exterior, and permanents in symmetric, algebras. From the definitions, the Schur scalar product generalized to $\Lambda^{\otimes}$, enjoys this crucial property with respect to the outer Hopf algebra:

Theorem 2.8. *The Schur scalar product is a $\mathbb{Z}$-valued Laplace pairing with respect to the outer product and coproduct:*

$$\begin{aligned} &(\{\lambda\} \mid \{\mu\} \cdot \{\nu\}) = (\{\lambda_{(1)}\} \mid \{\mu\})(\{\lambda_{(2)}\} \mid \{\nu\}) \\ &(\{\mu\} \cdot \{\nu\} \mid \{\lambda\}) = (\{\mu\} \mid \{\lambda_{(1)}\})(\{\nu\} \mid \{\lambda_{(2)}\}). \end{aligned}$$

Proof. This follows from the fact that the outer coproduct was introduced by duality from the outer product, and that (Δ, M) form a bialgebra (case I). □

Note that the corresponding property for inner product and coproduct does not constitute a Laplace pairing because of the lack of compatibility (table 1, case IV). From the coalgebra structures, cases I–III, more general Laplace properties can be inferred, for *non-scalar* pairings, given here for completeness:

Theorem 2.9.

(i) The skew product satisfies the (partial) Laplace conditions with respect to the outer and inner products:

$$\begin{aligned} &(\{\lambda\} \cdot \{\mu\})/\{\nu\} = (\{\lambda\}/\{\nu_{(1)}\}) \cdot (\{\mu\}/\{\nu_{(2)}\}) \\ &(\{\lambda\} \star \{\mu\})/\{\nu\} = (\{\lambda\}/\{\nu_{(1)}\}) \star (\{\mu\}/\{\nu_{(2)}\}). \end{aligned}$$

(ii) The inner product is a $\Lambda^{\otimes}$-valued Laplace pairing with respect to the outer product:

$$\begin{aligned} &\{\lambda\} \star (\{\mu\} \cdot \{\nu\}) = \sum (\{\lambda_{(1)}\} \star \{\mu\}) \cdot (\{\lambda_{(2)}\} \star \{\nu\}) \\ &(\{\lambda\} \cdot \{\mu\}) \star \{\nu\} = \sum (\{\lambda\} \star \{\nu_{(1)}\}) \cdot (\{\mu\} \star \{\nu_{(2)}\}). \end{aligned}$$

Proof. Part (i) refers to the outer Hopf algebra (case I) and the inner coproduct/outer product bialgebra (case II), interpreted for the dual action (2.20)—note, however, that $s_{\lambda/\mu} \neq s_{\mu/\lambda}$ in general. Part (ii) applies directly for the dual action from the outer coproduct–inner product bialgebra (case III). Finally, note that the inner coproduct–inner product convolution algebra does not admit the Laplace property. □

Of course, these formulae are known [32, 48, 2]. However, the above reasoning shows that they emerge from a single principle, which in turn generates Wick-like expansions (see [17, 18, 34], where such expansions are treated in detail; for the fermion–boson correspondence see [3] and [25, 23] for diagram strip decompositions and determinantal forms). The amazing computational power of the Laplace identities cannot be underestimated.

2.8. Kostka matrices

An immediate consequence of the observation that the Schur scalar product is a Laplace pairing is the fact, that it allows us to give a direct formula for the Kostka matrix in terms of the Littlewood–Richardson coefficients [35]. The Kostka matrix is defined as the *transition matrix M* from monomial functions to Schur functions $K = M(s, m)$. Since (m_λ) and (h_λ) are dual w.r.t. the Schur scalar product, and the Schur functions are self-dual, one obtains the transition matrix $K^* = M(s, h)$ for the basis change from complete symmetric functions to Schur functions. Noting that (*dot* is the outer product here) $h_n = s_{(n)}, h_\lambda = h_{\lambda_1} \cdot \ldots \cdot h_{\lambda_r}$, we compute (Sweedler indices as superscripts; note that (λ_i) is a one part partition and not a Sweedler index)

$$\begin{aligned}
K^*(s,h)_{\mu\lambda} &= (s_\mu \mid h_\lambda) \\
&= \big(s_\mu \,\big|\, s_{(\lambda_1)} \cdot \ldots \cdot s_{(\lambda_r)}\big) \\
&= \big(\Delta^{(r-1)}(s_\mu) \,\big|\, s_{(\lambda_1)} \otimes \cdots \otimes s_{(\lambda_r)}\big) \\
&= \sum \big(s_\mu^{(1)} \,\big|\, s_{(\lambda_1)}\big) \ldots \big(s_\mu^{(r)} \,\big|\, s_{(\lambda_r)}\big) \\
&= \sum_{\alpha_1,\ldots,\alpha_l} C^{\mu}_{(\lambda_1)\alpha_1} C^{\alpha_1}_{(\lambda_2)\alpha_2} \ldots C^{\alpha_{l-2}}_{(\lambda_{l-1})\alpha_l} \\
&\equiv \sum_\mu \prod_i \big(s_\mu^{(i)} \,\big|\, s_{(\lambda_i)}\big)
\end{aligned} \tag{2.40}$$

where the Littlewood–Richardson coefficients emerge from the coproduct, and $C^{\mu}_{\lambda(0)} = \delta^{\mu}_{\lambda}$ has been used.

Of course, similar calculations are possible for $M(h, s)$, $M(e, s)$, $M(s, e)$, etc. From Macdonald [35] (6.3)(3) one concludes further that if (u'), (v') are bases dual to (u), (v) then

$$M(u', v') = M(u, v)' = M(u, v)^* \tag{2.41}$$

holds, which shows that $K^*(s, h) = K(s, m)$ since $s' = s$.

Note that the above expansion can be used to compute the scalar product of the monomial and the complete symmetric functions in the following way,

$$\begin{aligned}
M(m,m)^*_{\lambda,\mu} &= M(h,h)_{\lambda\mu} = (h_\lambda \mid h_\mu) \\
&= \sum\sum\prod\prod \left(s^{(j)}_{(\lambda_i)} \,\middle|\, s^{(i)}_{(\mu_j)}\right).
\end{aligned} \tag{2.42}$$

The double sum and product is reminiscent of the fact that we have to expand both sides of the original scalar product. Especially interesting is the fact that

Lemma 2.10.

(i) $M(e, m)_{\lambda\mu} = \sum_\nu K_{\nu\lambda} K_{\nu'\mu}$ *is the number of matrices of 0 and 1 with row sums* λ_i *and column sums* μ_j.

(ii) $M(h, m)_{\lambda\mu} = \sum_\nu K_{\nu\lambda} K_{\nu\mu}$ *is the number of matrices of non-negative integers with row sums* λ_i *and column sums* μ_j.

Proof. Macdonald [35], (6.6)(i) and (ii) □

The Hopf algebraic expansion is done by using the fact that one can introduce a resolution of the identity,

$$\begin{aligned}
M(e,m) &= M(e,s)M(s,m) = M(s,m)M(h',s) \\
M(h,m) &= M(h,s)M(s,m) = M(s,m)M(h,s)
\end{aligned} \tag{2.43}$$

and using the above expansions.

3. Basic Hopf algebra cohomology

In this section we want to exploit the basic facts about Hopf algebra cohomology as developed by Sweedler [46]. Firstly we note the well-known theorem that

Theorem 3.1. *Every coassociative coalgebra map Δ induces face operators ∂_n^i and coboundary maps ∂_n from $H^{\otimes^n}$ to $H^{\otimes^{n+1}}$ as follows:*

$$\partial_n^i : H^{\otimes^n} \to H^{\otimes^{n+1}} \qquad \partial_n^i = 1 \otimes \cdots \otimes \Delta \otimes \cdots \otimes 1 \quad i\text{th place}$$

$$\partial_n : H^{\otimes^n} \to H^{\otimes^{n+1}} \qquad \partial_n = \sum_i \partial_n^i = \sum_i (-1)^i 1 \otimes \cdots \otimes \Delta \otimes \cdots \otimes 1 \quad i\text{th place}$$

$$\text{with} \quad \partial : H^{\otimes} \to H^{\otimes} \qquad \partial := \sum_n \partial_n \qquad \partial_{n+1} \circ \partial_n = 0. \tag{3.44}$$

Proof. An inspection of the face maps ∂_n^i shows that due to coassociativity, the coboundary maps ∂_n obey the desired relation, $\partial_{n+1} \circ \partial_n = 0$. □

In fact, we will not use this setting, but one pulled down from λ-ring addition to Hopf algebra convolution. Let $c_n \in \hom(H^{\otimes^n}, \mathbb{Z})$ be a normalized unital n-linear form, which we call n-cochain. Unitality is the property that $c(x_1 \otimes \cdots \otimes 1 \otimes \cdots \otimes x_n) = e$ for any occurrence of a unit. Since the λ-ring addition is given by the convolution—see the appendix—we will write the cohomology multiplicatively with respect to this outer convolution (M, Δ).

Definition 3.2. *The convolution product of two n-cochains c_n, c'_n is given as*

$$(c_n * c'_n)(x^1 \otimes \cdots \otimes x^n) = c_n\big(x^1_{(1)} \otimes \cdots \otimes x^n_{(1)}\big) c'_n\big(x^1_{(2)} \otimes \cdots \otimes x^n_{(2)}\big) \tag{3.45}$$

which is Abelian and associative. Define $c_n * c_m = 0$ if $n \neq m$. An n-cochain is assumed to be normalized $c_n(1 \otimes \cdots \otimes 1) = 1$, and hence known to be invertible $c_n^{-1} * c_n = \epsilon^{\otimes^n} = e$.

Aside. The inverse was given by Milnor and Moore for general homomorphisms under convolution by a recursive formula [37] (the inverse of the identity map $\mathrm{Id} : H^{\otimes} \to H^{\otimes}$ is by definition the antipode[13].) Since we will be able to use closed formulae for inverses in what follows, we need not take recourse in a computationally inefficient recursive definition.

We may now follow Sweedler and the development in [9] and define the coboundary operator acting on n-cochains in a multiplicative way w.r.t. convolution. See the appendix for the relation to λ-ring addition.

Definition 3.3.

$$\begin{aligned}
&\partial_n^0 c_{n-1}(x^1 \otimes \cdots \otimes x^n) := \epsilon(x^1) c_{n-1}(x^2 \otimes \cdots \otimes x^n)\\
&\partial_n^i c_{n-1}(x^1 \otimes \cdots \otimes x^n) := c_{n-1}(x^1 \otimes x^i x^{i+1} \cdots \otimes x^n)\\
&\partial_n^n c_{n-1}(x^1 \otimes \cdots \otimes x^n) := c_{n-1}(x^1 \otimes \cdots \otimes x^{n-1}) \epsilon(x^n)\\
&\partial_n c_n := \partial_n^0 c_n * \partial_n^1 c_n^{-1} * \partial_n^2 c_n * \cdots * \partial_n^n c_n^{\pm 1}\\
&\partial_{n+1} \circ \partial_n = e \qquad \text{with} \quad e := \epsilon \otimes \cdots \otimes \epsilon.
\end{aligned} \tag{3.46}$$

We will denote the coboundary map simply as ∂ if the context is clear.

Note that since we wrote the cohomology multiplicatively, the trivial element is the identity $e = \epsilon^{\otimes}$, the n-fold tensor product of the counit. Furthermore, the alternating sum of face

[13] The recursive formula of Milnor and Moore reduced to this case has been rediscovered recently in the context of QFT and is called the 'Connes–Kreimer' antipode formula.

maps $\sum \pm\partial_n^i$ translates into a 'see-saw' product of maps and inverse maps. We can use the coboundary map to classify n-cochains as follows:

Definition 3.4. *An n-coboundary $b_n \in B^n$ is an n-cochain fulfilling $b_n = \partial c_{n-1}$. An n-cocycle $c_n \in Z^n$ is an n-cochain fulfilling $\partial c_n = e$. An n-cochain which is neither a coboundary nor a cocycle will be called generic.*

It follows from the definition that an n-coboundary is an n-cocycle. Furthermore, the n-cocycles form an Abelian group under convolution. Hence, one may build the quotient of n-cocycles by n-coboundaries to form the nth cohomology group, $H^n = Z^n/B^n$. Indeed, this allows one to compute, e.g., the 'Betti' numbers of the complex (see [31]). From [9] we may take the characterization of 1-cocycles to be

Theorem 3.5. *A 1-cochain is a 1-cocycle if and only if it is an algebra homomorphism,*

$$\begin{aligned} e(s_\lambda \otimes s_\mu) = \partial c_1(s_\lambda \otimes s_\mu) = \sum_\alpha \sum_\beta c_1(s_\alpha) c_1(s_\beta) c_1^{-1}(s_{\lambda/\alpha} s_{\mu/\beta}) = \epsilon(s_\lambda)\epsilon(s_\mu) \\ \Leftrightarrow \quad c_1(s_\lambda s_\mu) = c_1(s_\lambda) c_1(s_\mu) \end{aligned} \tag{3.47}$$

where we have used the outer Hopf algebra in the convolution and exemplified the condition on a Schur function basis. This basic fact will suffice to make some observations and classifications in what follows related to the Schur function series introduced by Littlewood.

3.1. Littlewood–King–Wybourne infinite series of Schur functions

Littlewood [32] gave a set of Schur function series, which allowed him to formulate various identities in an extremely compact notation. These identities have been extended by King *et al* [27] and later by Yang and Wybourne [50]; we follow the presentation of the latter.

An S-function series is an infinite formal sum of Schur functions given via a generating function. It turns out that the most basic Schur function series is the so-called L-series,

$$L = \prod_{i=1}^{\infty} (1 - x_i) \tag{3.48}$$

from which the others may be derived. It is possible to give the S-function content of the series

$$L = \sum_{m=0}^{\infty} (-1)^m s_{(1^m)} = \sum_{m=0}^{\infty} (-1)^m \{1^m\} \tag{3.49}$$

where we have introduced the common notation $\{\lambda\}$ of Littlewood for a Schur function s_λ. Furthermore, it is convenient to follow Yang and Wybourne to introduce the conjugate (with respect to transposed partitions) series and the inverse conjugate series as

$$\begin{aligned} L^\dagger &= (\widetilde{L})^{-1} = \widetilde{L^{-1}} \\ &= \prod_{i=1}^{\infty} (1 + x_i)^{-1} = \sum_{m=0}^{\infty} (-1)^m \{m\}. \end{aligned} \tag{3.50}$$

Note that taking the conjugate is equivalent to the transformation $x_i \to -x_i$, which can be viewed as a plethysm (written with the concatenation symbol $\circ$ as before),

$$L = L(-x_i)^{-1} = (-\{1\}) \circ L^{-1}. \tag{3.51}$$

Table 2. *S*-function series: type, name = product, Schur function content and plethysm.

L	$L=\prod_i(1-x_i)$	$\sum_m(-1)^m\{1^m\}$	$L(x_i)$	$\{1\}\circ L$
L^{-1}	$M=\prod_i(1-x_i)^{-1}$	$\sum_m\{m\}$	$L(x_i)^{-1}$	$\{1\}\circ L^{-1}$
$\tilde{L}$	$P=\prod_i(1+x_i)^{-1}$	$\sum_m(-1)^m\{m\}$	$L(-x_i)^{-1}$	$(-\{1\})\circ L$
$L^\dagger$	$Q=\prod_i(1+x_i)$	$\sum_m\{1^m\}$	$L(-x_i)$	$(-\{1\})\circ L^{-1}$
A	$A=\prod_{i<j}(1-x_ix_j)$	$\sum_\alpha(-1)^{\omega_\alpha/2}\{\alpha\}$	$L(x_ix_j)(i<j)$	$\{1^2\}\circ L$
A^{-1}	$B=\prod_{i<j}(1-x_ix_j)^{-1}$	$\sum_\beta\{\beta\}$	$L(x_ix_j)^{-1}(i<j)$	$\{1^2\}\circ L^{-1}$
$\tilde{A}$	$C=\prod_{i\leqslant j}(1-x_ix_j)$	$\sum_\gamma(-1)^{\omega_\gamma/2}\{\gamma\}$	$L(x_ix_j)(i\leqslant j)$	$\{2\}\circ L$
$A^\dagger$	$D=\prod_{i\leqslant j}(1-x_ix_j)^{-1}$	$\sum_\delta\{\delta\}$	$L(x_ix_j)^{-1}(i\leqslant j)$	$\{2\}\circ L^{-1}$
$V=\tilde{V}$	$V=\prod_i(1-x_i^2)$	$\sum_{p,q}(-1)^p\{\widetilde{p+2q,p}\}$	$L(x_i^2)$	$(\{2\}-\{1^2\})\circ L$
$V^{-1}=V^\dagger$	$W=\prod_i(1-x_i^2)^{-1}$	$\sum_{p,q}(-1)^p\{p+2q,p\}$	$L(x_i^2)^{-1}$	$(\{2\}-\{1^2\})\circ L^{-1}$

For a Hopf algebraic approach to plethysm see [44]. The other series are then derived in a similar manner, see [50]. *S*-function series come in pairs which are mutually inverse and consecutively named. One finds

$$AB=1 \qquad CD=1 \qquad EF=1,\ldots \quad LM=1 \qquad PQ=1,\ldots \quad VW=1,\ldots \tag{3.52}$$

which may be arranged as in table 2, following [50]. The remaining series are $E=LA$, $F=L^{-1}A^{-1}$, $G=L^\dagger A$, $H=\tilde{L}A^{-1}$, $R=L\tilde{L}$ and $S=L^{-1}L^\dagger$. The partitions $\{\alpha\}$ associated with the A series are defined as follows (in Frobenius notation):

$$(\alpha)=(a_1a_2\cdots a_r \mid a_1+1a_2+1\cdots a_r+1) \tag{3.53}$$

and for the B, C and D series, $\{\beta\}$ is the transpose of $\{\alpha\}$, $\{\delta\}$ has only even parts and $\{\gamma\}$ is its transpose. $\{\epsilon\}$ (not in the table, but related to E) has only self-conjugate partitions and $\{\zeta\}$ (not in the table, but related to F) contains all partitions. Finally, ω_λ indicates the weight of the partition under consideration.

3.2. Cochain induced branching operators and series

The series in the above subsection play a fundamental role in the theory of group characters. We will show now, in which way these series are related with the action on Λ of a special class of endomorphisms[14]. In the following, we are interested in those endomorphisms of the ring of symmetric functions, which can be derived from 1-cochains. We introduce lower case symbols ϕ for 1-cochains, and the related operators $/\Phi:\Lambda\to\Lambda$, denoted by upper case letters, in the following way (cf (2.20)):

Definition 3.6. *An invertible branching operator is an endomorphism $/\Phi$ based on a 1-cochain ϕ via*

$$/\Phi(s_\lambda):=(\phi\otimes Id)\circ\Delta(s_\lambda)=\sum_\alpha\phi(s_\alpha)s_{\lambda/\alpha} \tag{3.54}$$

such that $\phi(s_\mu)\in\mathbb{Z}$.

We will now use the above-displayed results on cohomology to characterize the resulting maps.

[14] As will be discussed below, analogous endomorphisms also play a considerable role in QFT, where they are related to time, operator and normal ordering of quantum fields (see also [9]).

3.2.1. Skewing by a series. In group branching laws and formal S-function manipulations, one is interested in computing the skews of a particular irreducible representation described by a partition λ, with the elements of the series under consideration. Consider for example M,

$$s_\lambda/M = \sum_{m\in M} s_{\lambda/(m)} = s_{\lambda/(0)} + s_{\lambda/(1)} + s_{\lambda/(2)} + \cdots \tag{3.55}$$

where the resulting set of terms is actually finite, since $s_{\lambda/\mu}$ is zero if the weight of μ is greater than the weight of λ. In view of the outer coproduct,

$$\Delta(s_\lambda) = s_{\lambda/(0)} \otimes s_{(0)} + s_{\lambda/(1)} \otimes s_{(1)} + s_{\lambda/(1^2)} \otimes s_{(1^2)} + s_{\lambda/(2)} \otimes s_{(2)} + s_{\lambda/O(3)} \otimes s_{O(3)} \tag{3.56}$$

(where $O(3)$ means terms of weight equal or higher than 3) it is clear that a 1-cochain (linear form) can be defined to act on one tensor factor such that the resulting terms form the $/M$ skew series of s_λ. Generically for an arbitrary series Φ, the 1-cochain ϕ is the corresponding 'characteristic function',

$$\phi_\lambda \equiv \phi(s_\lambda) = \begin{cases} 1 & \text{if} \quad \lambda \in \Phi \\ 0 & \text{otherwise.} \end{cases} \tag{3.57}$$

This motivates *a posteriori* the name branching operator given to the above-defined operators $/\Phi$, and henceforth we adopt the notation $\Phi \cong /\Phi$ (see below).

Lemma 3.7. *The inverse branching operator is*

$$\Phi^{-1} = (\phi^{-1} \otimes Id) \circ \Delta \tag{3.58}$$

where the inverse of the 1-cochain ϕ is with respect to the outer convolution.

The outer product is trivial here since the value of the 1-cochain is in $\mathbb{Z}$.

Proof. We compute the composition of the two operators directly using coassociativity of the outer coproduct

$$\Phi^{-1}(\Phi(s_\lambda)) = \sum_{\alpha,\beta} \phi^{-1}(s_\alpha)\phi(s_{\beta/\alpha})s_{\lambda/\beta} = \sum_\beta \epsilon(s_\beta)s_{\lambda/\beta} = s_\lambda. \tag{3.59}$$

□

Thus the obvious inverse operation, skewing by the inverse series (with respect to the outer product), has an internal structure governed by outer convolution at the level of the underlying 1-cochain. Finally, this allows us to form the following new product M_ϕ, which will be set in a more general context in the next section.

Definition 3.8. *The ϕ-deformed outer product M_ϕ is defined as*

$$\mathsf{M}_\phi(f \otimes g) = f \circ_\phi g = \Phi^{-1}(\mathsf{M}(\Phi(f) \otimes \Phi(g))). \tag{3.60}$$

3.2.2. Classifying branching operators. From cohomology we know already that there are cochains of different types, for example, 1-cocycles and generic 1-cochains (there are no 1-coboundaries). Hence, we expect that this difference shows up in the nature of the branching operators induced by these 1-cochains.

Trivial 1-cochain. The trivial 1-cochain is the counit. By definition, the counit acts such that the coproduct action is void, and we get as branching operator for the trivial 1-cochain, the identity.

1-cocycles. From the cohomology we learned that the property of being a 1-cocycle ϕ is equivalent to being an algebra homomorphism, $\phi(\{\lambda\} \cdot \{\mu\}) = \phi(\{\lambda\})\phi(\{\mu\})$, alternatively written as $(s_\lambda \cdot s_\mu)/\Phi = s_\lambda/\Phi \cdot s_\mu/\Phi$. The same is true for the inverse 1-cocycle ϕ^{-1}. This allows us to state that

Lemma 3.9. *The outer product* M *and the* ϕ*-deformed outer product* M_ϕ *are isomorphic if and only if* ϕ *is a 1-cocycle.*

Proof. This follows directly from the definition, and the fact that the inverse branching operator is also an algebra homomorphism. □

Hence we find

$$\phi \text{ is a 1-cocycle} \quad \Leftrightarrow \quad \phi \in \text{alg-hom}((\Lambda^{\otimes}, M), (\Lambda^{\otimes}, M)). \tag{3.61}$$

As we will see later, these deformed products are far from being empty constructs. Indeed, we have a coalgebra action and an augmentation, the counit ϵ, and one checks easily that

Lemma 3.10. *The augmented outer (comodule) algebras* $(\Lambda^{\otimes}, \mathsf{M}, \epsilon)$ *and* $(\Lambda^{\otimes}, \mathsf{M}_\phi, \epsilon)$ *are non-isomorphic.*

Proof. Since the counit is *not* transformed, it acts differently on the two algebras. □

Indeed, the *isomorphic* structures are related by adopting as the ϕ-deformed counit

$$\epsilon \to \epsilon \circ \Phi = \phi * \epsilon \equiv \phi \tag{3.62}$$

which is in general different from ϵ. This may be worth exemplification, so for a general Φ we compute the action of ϵ on $s_{(1)} \cdot s_{(1)}$ and $s_{(1)} \circ_\phi s_{(1)}$:

$$(a)\ \epsilon(s_{(1)} \cdot s_{(1)}) = \epsilon(s_{(2)} + s_{(1^2)}) = 0$$
$$\begin{aligned}(b)\ \epsilon(\Phi^{-1}(\Phi(s_{(1)}) \cdot \Phi(s_{(1)}))) &= \epsilon \circ \Phi^{-1}((s_{(1)}\phi_0 + s_0\phi_{(1)}) \cdot (s_{(1)}\phi_0 + s_0\phi_{(1)})) \\ &= \epsilon \circ \Phi^{-1}(s_{(1)} \cdot s_{(1)} + 2\phi_{(1)}s_{(1)} + {\phi_{(1)}}^2) \\ &= {\phi_{(1^2)}}^{-1} + {\phi_{(2)}}^{-1} + {\phi_{(1)}}^2 \end{aligned} \tag{3.63}$$

which is $\neq 0$ in general. (In the last line the normalization ${\phi_0}^{-1} = \phi_0 = 1$ and the definition of ϵ has been used).

Generic 1-cochain. For generic 1-cochains the above consideration fails, hence we can only state that a deformed outer product M_ϕ based on a generic 1-cochain is non-isomorphic to the original outer product. The relation between these two products will become clear in the next section.

3.2.3. Classifying series. In summary, the cohomological properties of the 1-cochains classify the associated branching operators Φ. Besides the identity, we get those based on 1-cocycles which produce isomorphic products, and generic ones. Since we *defined* the S-function series using branching operators, we can in turn classify the series into three families according to the underlying cochains: those based on coboundaries (empty for 1-cochains), those based on cocycles and those based on generic cochains.

Theorem 3.11. *The series derived from branching operators based on 1-cocycles* ϕ *fulfil*

$$\{\lambda \cdot \mu\}/\Phi = \{\lambda\}/\Phi \cdot \{\mu\}/\Phi \tag{3.64}$$

reflecting the homomorphism property.

Proof. Note that $/\Phi$ was defined as $(\phi \otimes \mathrm{Id})\circ\Delta$. Using the product–coproduct homomorphism axiom of bialgebras we compute

$$(\phi \otimes \mathrm{Id})(\Delta \circ M) = (\phi \otimes \phi \otimes M)(\mathrm{Id} \otimes \mathsf{sw} \otimes \mathrm{Id})(\Delta \otimes \Delta) \tag{3.65}$$

which, after reformulating using $/\Phi$ yields the assertion. □

It is now a matter of explicit calculation to check which series fulfil the cocycle property. The generic case (for the remaining series) comes up in two different forms depending on the appearance of the antipode. One finds

Theorem 3.12.

(i) Branching operators based on the series L, M, P, Q, R, S, V, W fulfil (3.64).

(ii) Branching operators based on generic 1-cochains of the B, D, F, H series satisfy

$$\{\lambda \cdot \mu\}/\Phi = \sum_\zeta \{\lambda\}/(\zeta \cdot \Phi) \cdot \{\mu\}/(\zeta \cdot \Phi). \tag{3.66}$$

(iii) Branching operators based on generic 1-cochains of the A, C, E, G series satisfy[15]

$$\begin{aligned}\{\lambda \cdot \mu\}/\Phi &= \sum_\zeta \{\lambda\}/(\zeta \cdot \Phi) \cdot \{\mu\}/(\mathsf{S}(\zeta) \cdot \Phi) \\ &= \sum_\zeta (-1)^{\omega_\zeta} \{\lambda\}/(\zeta \cdot \Phi) \cdot \{\mu\}/(\zeta' \cdot \Phi).\end{aligned} \tag{3.67}$$

where the summation is over all Schur functions ζ (i.e. using the F series).

Proof. The proof and further material in this direction can be found in [6, 27, 28]. A direct proof is based on formal evaluation of the outer coproduct, based on (2.14). For the series D (table 2) we have, for example

$$\begin{aligned}D(x, y) &= \prod_{i\leqslant j}(1 - x_i x_j)^{-1} \prod_{\ell\leqslant m}(1 - y_k y_\ell)^{-1} \prod_{k,n}(1 - x_k y_n)^{-1} \\ &= D(x)D(y) \sum_\zeta s_\zeta(x) s_\zeta(y) \\ &= \sum_\zeta D(x)s_\zeta(x) \cdot D(y)s_\zeta(y)\end{aligned}$$

where the fundamental Cauchy identity has been used in the second line [35]. □

We shall call the series L, M, P, Q, R, S, V, W *group-like*, because of the above outer coproduct property and in view of (3.64) (see below). In fact, it is easy to see that the above argument goes through for *any* series defined by a generating function of the form $\Phi = \prod_i (1 - f(x_i))^s$ for some polynomial $f(x)$. Note finally that the second two lists are made from mutually inverse elements A, B; C, D; E, F and G, H.

3.3. Products versus orderings

It is worth noting at this stage that a structural analogy in relation to products and operator ordering can be seen with quantum physics. Just as there both normal and time ordered products are needed, so too in symmetric function theory different products induced by branching operators emerge: the rings (Λ, M) and $(\Lambda, \mathsf{M}_\phi)$ related by a cocycle are ϕ-isomorphic (but differ under the augmentation by the co-unit, ϵ)[16]. In order to derive further Hopf-algebraic

[15] Recall that ζ' is the partition conjugate to ζ.

[16] This analogy is elaborated in the concluding remarks below in terms of an *active* versus *passive* analysis of the role of the branching endomorphisms.

motivated insights into these formulae, we need to consider 2-cochains, and reconsider the Schur scalar product and its convolutive inverse in a more elaborated approach to deformed products, namely that of cliffordization.

4. Cliffordization

4.1. Definition

The first occurrence of a cliffordization is to our knowledge in Sweedler [46], where it was derived from a smash product in a cohomological context. Later, Drinfeld [13] invented the twisting of products, also induced by a Hopf algebraic construction related to a smash product. The term 'Cliffordization' was coined in [40, 41]. There a coalgebra structure was also employed, but not in general a Hopf algebra, and the combinatorial aspects were emphasized. Since we follow the direction taken by Rota in the treatment of symmetric functions, it seems reasonable to stick to this technical term. A more Hopf-algebraic motivated approach is included in [9].

Let us assume that we start with a Hopf algebra, but focus our interest for the moment on the algebraic part of it. Given a linear space underlying an algebra, we are interested to induce a family of deformed products using a pairing. Special properties of the pairing will ensure special properties of the deformed product. In general, such a pairing should be a measuring, to keep the Hopf algebraic character of the whole structure [45]. However, we will drop this requirement. The resulting structure is a comodule algebra, hence an algebra with a coaction, which is not necessarily a bialgebra or Hopf.

Definition 4.1. *A self-pairing is a linear map $\pi : \Lambda^{\otimes} \otimes \Lambda^{\otimes} \to \Lambda^{\otimes}$. A scalar pairing has $\mathbb{Z}$ as codomain.*

We will be interested in scalar self-pairings. Indeed, we have already made use of the 'natural' pairing of symmetric functions, that is, the Schur scalar product. This allows us to give the following:

Definition 4.2. *A cliffordization is a deformation of a comodule, or possibly a Hopf, algebra $(\Lambda, \cdot)$ into a twisted comodule algebra on the same space Λ equipped with the circle product*

$$x \circ y = \sum \pi(x_{(1)} \otimes y_{(1)}) x_{(2)} \cdot y_{(2)}.$$

The name cliffordization[17] stems from the fact that if π is a 'scalar product' and $(V^{\wedge}, \wedge)$ a Grassmann algebra, then $\circ$ is the endomorphic Clifford product induced by π [18, 19].

In the case of Schur functions a twisted or cliffordized product can be given by the Schur scalar product *or* its inverse playing the role of the pairing π. From Hopf algebra theory we know further that the inverse of a 2-cocycle, w.r.t. the convolution, is given by acting with the antipode in the first or second argument,

$$\pi(x \otimes y) = \pi(\mathsf{S}(x) \otimes \mathsf{S}(y)) \qquad \pi(x \otimes y)^{-1} = \pi(\mathsf{S}(x) \otimes y) = \pi(x \otimes \mathsf{S}(y)) \tag{4.68}$$

which allows the introduction of the (off-diagonal) convolutive *inverse* Schur scalar product in terms of the transpose as

$$(s_\lambda \mid s_\mu)^{-1} = (-1)^{\omega_\mu}(s_\lambda \mid s_{\mu'}). \tag{4.69}$$

[17] The 'circle product' is *not* to be confused with plethysm, which will play no role for the moment.

Thus the scalar product cliffordizations read

$$
\begin{aligned}
s_\lambda \circ s_\mu &= \sum_{\alpha,\beta} (s_\alpha \mid s_\beta) s_{\lambda/\alpha} \cdot s_{\mu/\beta} \\
&= \sum_{\alpha} s_{\lambda/\alpha} \cdot s_{\mu/\alpha} \\
s_\lambda \circ_{\mathsf{S}} s_\mu &= \sum_{\alpha,\beta} (-1)^{\omega_\beta} (s_\alpha \mid \mathsf{S}(s_\beta)) s_{\lambda/\alpha} \cdot s_{\mu/\beta} \\
&= \sum_{\alpha} (-1)^{\omega_\alpha} s_{\lambda/\alpha} \cdot s_{\mu/\alpha'}
\end{aligned}
\tag{4.70}
$$

since the Schur functions are orthogonal. Clearly the two formulae in (4.70) and those in (3.66) and (3.67) are based on the Schur scalar product and its inverse, up to the additional appearance of a series. The situation is summarized in the following section.

4.2. Classifying 2-cochains and cliffordization

Scalar pairings can be regarded as 2-cochains. It is therefore convenient to classify them in analogy with the 1-cochains. We find the following:

Trivial 2-cochain. The trivial 2-cochain is the map $e^{\otimes^2} = \epsilon \otimes \epsilon$. Hence in substituting $\pi = e^{\otimes^2}$ in (4.70) one sees that the product just remains unaltered, and this yields the identity deformation.

2-coboundaries. A 2-coboundary is a pairing which is derived from a 1-cochain via the coboundary operator, $\pi = \partial\phi$ for a 1-cochain $c_1 \equiv \phi$. Looking at (3.46) for the case $n = 2$ (see (3.47) for the expansion of the convolution products in Sweedler notation) we see that these 2-cochains are of the form

$$
\pi_\phi(x \otimes y) = \partial\phi(x \otimes y) = \phi(x_{(1)})\phi(y_{(1)})\phi^{-1}(x_{(2)}y_{(2)}). \tag{4.71}
$$

This shows that the *group-like* deformations by a series can be equally well addressed as a cliffordization by a 2-coboundary. In fact, one is able to rearrange the ϕ-deformed outer product in the form

$$
\mathsf{M}_\phi(x \otimes y) = \Phi^{-1}(\mathsf{M}(\Phi(x) \otimes \Phi(y)) \equiv \sum \pi_\phi(x_{(1)} \otimes y_{(1)}) x_{(2)} \cdot y_{(2)} \tag{4.72}
$$

if and only if the 2-cochain π is a coboundary [18].

2-cocycles. A 2-cocycle is characterized by $\partial\pi = e$. It is well known from Hopf algebra theory, and from the theory of $*$-product deformations that deformations induced by a 2-cocycle yield associative products (for details see [9] and references therein). Moreover, since our 2-cochains are assumed to be normalized, they are invertible. One finds

$$
(\circ_\phi)_{\phi^{-1}} = \circ_{\phi * \phi^{-1}} = \circ_e \tag{4.73}
$$

showing the invertibility of the deformation process. For the case of symmetric functions and the Schur scalar product and its inverse, the cliffordizations are (4.70), as already noted above.

Generic 2-cochains. The deformed circle product based on a generic 2-cochain cannot be associative and we will here not consider such deformations.

Mixed 2-cocycles and 2-coboundaries. It is possible to draw the analogy that 2-coboundaries are topological 'gauges', while the 2-cocycles are generic topological 'fields'. Since the space of cocycles, boundary or not, forms a group, we can pick a section in the orbits of 2-coboundaries, i.e. gauge the 2-cocycles by adding a 2-coboundary as

$$\pi = c_2 * \partial c_1. \tag{4.74}$$

Taking one of the formulae from (4.70) and a 2-coboundary which we know to be derived from a 1-cochain, that is from a series, we have

$$\begin{aligned} x \circ_\phi y &= \sum ((\cdot \mid \cdot) * \partial_\phi)(x_{(1)} \otimes y_{(1)}) x_{(2)} \cdot y_{(2)} \\ x \circ_{\mathsf{S},\phi} y &= \sum ((\cdot \mid \mathsf{S}(\cdot)) * \partial_\phi)(x_{(1)} \otimes y_{(1)}) x_{(2)} \cdot y_{(2)} \end{aligned} \tag{4.75}$$

which can be rewritten in the form analogously to (3.66) and (3.67) (for the generic cases of theorem 3.12),

$$\begin{aligned} (s_\lambda \cdot s_\mu)/\Phi &= \sum (s_\alpha \mid s_\beta) s_{\lambda/(\alpha\Phi)} \cdot s_{\mu/(\beta\Phi)} \\ &= \sum s_{\lambda/(\alpha\Phi)} \cdot s_{\mu/(\alpha\Phi)} \\ (s_\lambda \cdot s_\mu)/_{\mathsf{S}}\Phi &= \sum (s_\alpha \mid \mathsf{S}(s_\beta)) s_{\lambda/(\alpha\Phi)} \cdot s_{\mu/(\beta\Phi)} \\ &= \sum (-1)^{\omega_\alpha} s_{\lambda/(\alpha\Phi)} \cdot s_{\mu/(\alpha'\Phi)} \end{aligned} \tag{4.76}$$

where we have used (4.71) and $(s_\lambda/\alpha)/\Phi = s_{\lambda/(\alpha\Phi)}$. Hence we proved the following two lemmas:

Lemma 4.3. *Equations (3.66) and (3.67) are cliffordizations w.r.t. the Schur scalar product or its inverse, in convolution with a 2-coboundary induced by an S-function series (or branching operator). These formulae are precisely the Newell–Littlewood products (see below).*

Lemma 4.4. *The two instances of the formulae in (3.66) and (3.67) are mutually inverse to one another if the involved series are mutually inverse, e.g., A, B; C, D; etc.*

4.3. Eight possible Cliffordizations

To finish the technical parts of the discussion of cliffordization, we want to present an overview on what kind of open possibilities remain to be explored. Indeed, looking at the stock of ‘natural’ structures in symmetric function theory, we find the Schur scalar product and its inverse, the outer and inner products and the outer and inner coproducts. Examining the definition of cliffordization, one notes that it involves two coproducts, one 2-cocycle and 1 product. If we consider the inverse Schur scalar product not as essentially different from the Schur scalar product, then we find a total of eight different possibilities to employ the inner and outer products and coproducts in cliffordization. We obtain the cliffordizations and the grades for products of homogenous elements as

$$\begin{aligned} f \circ_1 g &= \sum \pi(f_{(1)} \otimes g_{(1)}) \mathsf{M}(f_{(2)} \otimes g_{(2)}) & |n| \otimes |m| &\to \oplus_r |n+m-2r| \\ f \circ_2 g &= \sum \pi(f_{(1)} \otimes g_{(1)}) \mathsf{m}(f_{(2)} \otimes g_{(2)}) & |n| \otimes |m| &\to \oplus_r \delta_{nm} |n-r| \\ f \circ_3 g &= \sum \pi(f_{[1]} \otimes g_{(1)}) \mathsf{M}(f_{[2]} \otimes g_{(2)}) & |n| \otimes |m| &\to |m| \\ f \circ_4 g &= \sum \pi(f_{[1]} \otimes g_{(1)}) \mathsf{m}(f_{[2]} \otimes g_{(2)}) & |n| \otimes |m| &\to \delta_{n,m-n} |n| \\ f \circ_5 g &= \sum \pi(f_{(1)} \otimes g_{[1]}) \mathsf{M}(f_{(2)} \otimes g_{[2]}) & |n| \otimes |m| &\to |n| \\ f \circ_6 g &= \sum \pi(f_{(1)} \otimes g_{[1]}) \mathsf{m}(f_{(2)} \otimes g_{[2]}) & |n| \otimes |m| &\to \delta_{n-m,m} |m| \\ f \circ_7 g &= \sum \pi(f_{[1]} \otimes g_{[1]}) \mathsf{M}(f_{[1]} \otimes g_{[1]}) & |n| \otimes |m| &\to \delta_{nm} |2n| \\ f \circ_8 g &= \sum \pi(f_{[1]} \otimes g_{[1]}) \mathsf{m}(f_{[1]} \otimes g_{[1]}) & |n| \otimes |m| &\to \delta_{nm} |n| \end{aligned} \tag{4.77}$$

where we used the Brouder–Schmitt convention on coproducts and M, m for the outer and inner product of symmetric functions. The right column displays the grades obtained in multiplying two homogenous elements of grades $|n|$ and $|m|$. Of course, any 2-cocycle induces in this way eight cliffordizations. This raises the question about a classification of all 2-cocycles acting on the ring of symmetric functions $\Lambda^{\otimes^2}$. While we have considered the Schur scalar product for the ring $\otimes_{\mathbb{Z}}\Lambda$, the appearance of Hall–Littlewood symmetric functions and Macdonald symmetric functions clearly shows that these cases are tied to ring extensions. From Scharf and Thibon's approach to inner plethysm [44] (p 33) it is obvious that the change of one structure map in a convolution changes its properties dramatically. One finds

$$\begin{aligned} f +_\Lambda g \quad &\Leftrightarrow \quad \mathsf{M} \circ (f \otimes g) \circ \Delta \\ f \cdot_\Lambda g \quad &\Leftrightarrow \quad \mathsf{M} \circ (f \otimes g) \circ \delta. \end{aligned} \tag{4.78}$$

Hence changing the coproduct from outer to inner changes the λ-ring operation induced by convolution from addition to multiplication (see the appendix). In fact, this is the source of the difference between the outer and inner branching rules. This observation gives a hint, as to the way in which the above cliffordizations may change if inner products and coproducts replace the outer products and coproducts. A few quite amazing properties can be easily derived, but we will not enter this subject here.

4.4. Branching rules: $U(n) \downarrow O(n)$; $U(n) \downarrow Sp(n)$; $O(n) \uparrow U(n)$; $Sp(n) \uparrow U(n)$ and product rules: $Sp(n) \times Sp(n) \downarrow Sp(n)$, $O(n) \times O(n) \downarrow O(n)$

The above-discussed relations for Schur functions reflect their interpretation as universal characters [29]. We will not deal with the problem of modification rules needed for actual evaluation of the reduced group characters, but follow in our presentation King *loc cit*. Our aim thereby is to make clear in this section the connection between the Hopf algebraic approach and the group theory. The basic starting point is Weyl's character formula

$$\mathrm{ch}(\Lambda) = \sum_{w \in W} \epsilon(w)\, \mathrm{e}^{w(\Lambda+\rho)} \Big/ \sum_{w \in W} \epsilon(w)\, \mathrm{e}^{w(\rho)} \tag{4.79}$$

where Λ is the highest weight vector, ρ is half the sum of the positive roots and W is the appropriate Weyl group with ϵ the sign of w. The Cartan classification of the simple complex classical Lie groups is given by the series A_n, B_n, C_n and D_n, not to be confused with Schur function series. They correspond to the complexified versions of the groups $SU(n+1)$, $SO(2n+1)$, $Sp(2n)$ and $SO(2n)$. These groups can be considered as subgroups of unitary groups $U(N)$ for $N = n+1, 2n+1, 2n$ and $2n$. Denoting eigenvalues as $x_k = \exp(\mathrm{i}\phi_k)$ and $\bar{x}_k = x_k^{-1}$ one can write the eigenvalues of group elements in the following way:

$$\begin{array}{lll} SU(n+1) & x_1, x_2, \ldots, x_n & \text{with} \quad x_1 x_2 \cdots x_n = 1 \\ SO(2n+1) & x_1, x_2, \ldots, x_n, \bar{x}_1, \bar{x}_2 \cdots \bar{x}_n, 1 & \\ Sp(2n) & x_1, x_2, \ldots, x_n, \bar{x}_1, \bar{x}_2 \cdots \bar{x}_n & \\ SO(2n) & x_1, x_2, \ldots, x_n, \bar{x}_1, \bar{x}_2 \cdots \bar{x}_n. & \end{array} \tag{4.80}$$

The connection to the group characters is obtained by inserting the eigenvalues into the Weyl character formula and interpreting the exponentials as

$$e^{\lambda} = x_1^{\lambda_1} x_2^{\lambda_2} \cdots x_n^{\lambda_n} \qquad \lambda = (\lambda_1, \ldots, \lambda_n). \tag{4.81}$$

In the case of $U(n)$, the Weyl group is just the symmetric group (on n letters). Hence, the characters are labelled by partitions and the Weyl character formula turns into the defining relation of the Schur functions. Let μ be the conjugacy class of the permutation. One finds

$$\mathrm{ch}_\mu(\lambda) = \sum_{w\in S_n} \epsilon(w)\, \mathrm{e}^{w(\lambda+\rho).\mu} \Big/ \sum_{w\in S_n} \epsilon(w)\, \mathrm{e}^{w(\rho.\mu)} \tag{4.82}$$

where $\rho = (n-1, n-2, \ldots, 1, 0)$. Both numerator and denominator reduce to determinants after inserting the x_i (the denominator being the van der Monde determinant), and the quotient of the two alternating functions is a standard construction of the Schur function [35].

Let us introduce the standard notation for group characters

$$\begin{array}{ll} U(n) & \mathrm{ch}_\mu(\lambda) = \{\lambda\}(x)_n \\ O(n) & \mathrm{ch}_\mu(\lambda) = [\lambda](x)_n \\ Sp(n) & \mathrm{ch}_\mu(\lambda) = \langle\lambda\rangle(x)_n. \end{array} \tag{4.83}$$

It is well known that one has the following relations using S-function series:

$$\begin{array}{ll} U(n) \downarrow O(n) & \{\lambda\}(x) = [\lambda/D](x) \\ O(n) \uparrow U(n) & [\lambda](x) = \{\lambda/C\}(x) \\ U(n) \downarrow Sp(n) & \{\lambda\}(x) = \langle\lambda/B\rangle(x) \\ Sp(n) \uparrow U(n) & \langle\lambda\rangle(x) = \{\lambda/A\}(x). \end{array} \tag{4.84}$$

In fact, this justifies the name *branching operator* for the operators which we had defined in (3.54), at least in the cases A, B, and C, D. Looking at the table (3.53) that A, B are derived from the basic L series by a plethysm with $\{1^2\}$, reflecting the fact that symplectic groups have antisymmetric bilinear forms as metric, while C, D are related to the plethysm of the L series with $\{2\}$, reflecting this time the fact that orthogonal groups are based on symmetric bilinear forms.

We noted above that formula (3.64) is valid for generating functions of the form $\prod(1 \pm f(x_i))$, for polynomial f. This was the origin of these series being group-like, and could be tied to a 1-cocycle ensuring that the 'branching operator' gave an algebra homomorphism. In this sense only a trivial branching process is involved (although the transformation involved could be very complicated in detail). As an example, one might look at the series V, W in table (2), which being plethysms by $(\{2\} - \{1^2\}) \equiv p_2$ are group-like, since $V = L\big(x_i^2\big)$, $W = L\big(x_i^2\big)^{-1}$. This may be summarized in the statement

Lemma 4.5. *All group-like series $\Phi = \prod_i (1 - f(x_i))^s$ (based on 1-cocycles) induce trivial branchings, i.e. branchings equivalent to $U(n)$.*

We noted above, that 1-cocycles cannot induce non-trivial 2-cocycles, since by definition $\partial c_1 = e$. For a product deformation we would need a non-trivial 2-cocycle, at least a 2-coboundary, hence this outcome is in full accord with our cohomological classification.

Finally, the non-trivial series not based on 1-cocycles are no longer algebra homomorphisms, and one cannot expect that the branching law for characters remains valid. Thus the A, B and C, D series are not group-like. However, the defect from being a homomorphism is fully compensated by the cliffordization w.r.t. the Schur scalar product or its inverse, as shown in (3.66), (3.67), and (4.76), respectively. This is a well-known fact and describes the product rules of the groups $Sp(n)$ and $O(n)$, which can equally be seen as the branching rules $Sp(n) \times Sp(n) \downarrow Sp(n)$, $O(n) \times O(n) \downarrow O(n)$:

Theorem 4.6. (Newell–Littlewood [38, 33])

$$\begin{aligned} Sp(n) \qquad & \langle\lambda\rangle \otimes \langle\mu\rangle = \sum_{\zeta} \langle(\lambda/\zeta)\cdot(\mu/\zeta)\rangle \\ O(n) \qquad & [\lambda] \otimes [\mu] = \sum_{\zeta} [(\lambda/\zeta)\cdot(\mu/\zeta)] \end{aligned} \tag{4.85}$$

where $\cdot$ is the outer product of S-functions, / the S-function quotient and $\otimes$ is the Kronecker tensor product[18] *of the universal characters.*

In fact, we found in lemma 4.4 that the inverse of the branchings given by the Newell–Littlewood theorem is given by the cliffordized products $\circ_{(.|\mathsf{S}(.))*\Phi^{-1}}$, where the inverse series is employed, but also the convolutive inverse Schur scalar product $(. \mid \mathsf{S}(.))$. In this way, the cliffordization w.r.t. the Schur scalar product provides the compensation for the fact that the series acting in the branching are not homomorphisms. This reads explicitly

$$\begin{aligned} f \circ_{(.|.)*\Phi} g &= \Phi^{-1}(\Phi(f) \circ_{(.|.)} \Phi(g)) \\ &= \Phi^{-1} \sum (\Phi(f_{(1)}) \mid \Phi(g_{(1)}))\Phi(f_{(2)}) \cdot \Phi(g_{(2)}). \end{aligned} \tag{4.86}$$

This beautiful result immediately raises the following questions, among others. Is it possible to define other generic cliffordizations, i.e. inner products which are 2-cocycles, and what are the corresponding series? Also, while it is clear that series such as $L\left(x_i^3\right)$ are still group-like, it seems to be questionable if series with three or more independent variables can lead to an associative multiplication, i.e. are based on a 2-cocycle. Hence the question, what kind of 'branching rule' can be derived for series of the form say $\{3\} \circ L$, $\{21\} \circ L$ or $\{1^3\} \circ L$? In fact one awaits no 'group' here, since all classical groups are well known and exhausted by the above cases.

5. Conclusions and analogy with quantum field theory

In this paper we have given a synthesis of aspects of symmetric function theory from the viewpoint of underlying Hopf- and bi-algebraic structures. The focus has been on the explicit presentation of basic definitions and properties satisfied by the fundamental ingredients—outer and inner products and coproducts, units and counits, outer antipode, Schur scalar product, skew product—with ramifications for Laplace pairings, Kostka matrices, Sweedler cohomology and especially Rota cliffordizations. Our main result is that there is a rich class of associative deformations isomorphic to the standard outer product (but non-isomorphic as augmented algebras), and that the Newell–Littlewood product for symmetric functions of orthogonal and symplectic type is an associative deformation non-isomorphic to the outer product, derived from a 2-cocycle, up to a coboundary in the Sweedler sense. The conclusion of our analysis is the recognition that, even at this level, many familiar constructs from the rich theory of symmetric functions can be encapsulated by the organizing power of the co-world of Hopf- and bi-algebras.

No attempt has been made to extend the analysis beyond the standard symmetric polynomials–Hall–Littlewood, Macdonald, Jack, Kerov, MacMahon/vector, factorial/hypergeometric, ... symmetric functions should enter the Hopf framework at points where the structure admits natural generalizations. For example, while we have considered

[18] The notation $\circ_{(|)*B}$ and $\circ_{(|)*D}$ for the cliffordized products is perhaps more suggestive.

the Schur scalar product for the ring $\otimes_{\mathbb{Z}}\Lambda$, the appearance of Hall–Littlewood and Macdonald symmetric functions shows clearly the necessity for ring extensions and for the formerly symmetric monoidal category to turn into a braided one. Thus, q-Laplace expansions (in the context of a q-braided crossing) should play a crucial role in unveiling the nature of q-Kostka matrices and q-Littlewood–Richardson coefficients. Further extensions of $\mathbb{Q}$ by irrationals obtain from evaluating the q in the q-polynomials. Such a ring $\mathbb{Q} \otimes_{\mathbb{Z}[v_1,v_2,\ldots]} \Lambda$ has a much more interesting cohomology group and therefore should bear a rich class of possible different cliffordizations (with non-cohomologous 2-cocycles).

Finally a disclaimer should be made that in the present work no attempt has been made to address deeper issues of symmetric function theory such as outer and inner plethysms, the role of vertex operators and the fermion–boson correspondence. For Hopf algebra approaches to plethysm see [44]; the Littlewood–Richardson rule and the fermion–boson correspondence is discussed in [3]. For matrix elements of vertex operators using composite supersymmetric S-functions see [25] and [4] for vertex operators for symmetric functions of orthogonal and symplectic type. Relations to inner plethysm are given in [43].

Several parallels between symmetric functions and combinatorial approaches to QFT have been alluded to in the text. We conclude with an amplification of these points, which may provide further motivation for the programme outlined here (see also [9]). An underlying cornerstone in combinatorics and in its application to QFT is the following: given a set of objects, called letters (or 'balls' if we use a combinatorial notion), one is interested in the first instance in the relation of these objects (putting (weighted) balls together into boxes). This level is given by the symmetric functions, or $\mathsf{Tens}[L]$ where L is the letter 'alphabet', hence the variables $\{x_i\}$ (the balls). The *grading* of the tensor algebra imposes collections inhabited by symmetric functions of n-variables (putting n-balls into boxes). Having objects and morphisms, we are ready to form a category. In a second step, we are dealing with deformations of operations, which live in $\mathsf{EndTens}[L]$. This is asking for operations on operations, or more physically speaking 'parametrization' of operations (putting boxes into packages). Mathematically speaking we are dealing with a 2-category. In the theory of symmetric functions, Schur functions are used as symmetric functions, hence $s_\lambda \in \mathsf{Tens}[L]$ (boxes with balls) *and at the same time* as polynomial functors (see [35] chapter I, appendix), i.e. as operators *on* symmetric functions (packages collecting boxes). It was to our knowledge Gian-Carlo Rota who made this explicit. Hence we are dealing with symmetric functions from $\Lambda(X) \cong \mathsf{Tens}[L]$ *and* with endomorphisms living in $\Lambda^{\otimes} \cong \Lambda(X, Y, \ldots) \cong \mathsf{TensTens}[L]$. The process of moving up one step in a hierarchy or stack of categories, i.e. moving from 1-categories to 2-categories (to n-categories) is *categorification* [5].

The triply iterated structure alluded to here is perfectly mirrored in the process of second quantization (coordinates $\to$ wavefunctions $\to$ functionals), and supports the suggestion that some of the machinery of quantum field theory can be captured at the combinatorial level. Recall for instance the role of Kostka matrices (lemma 2.10) in counting column and row sum matrices. As mentioned, in quantum field theory the *fields*[19] have to be considered as the variables 'x', and hence the double Kostka matrices $M(h, m)$ or $M(e, m) = [a_{ij}]$ appear as exponents in expressions such as

$$\sum\sum\prod\prod(\phi_i(x)|\phi_j(y))^{a_{ij}} \tag{5.87}$$

where the a_{ij} are from $M(e, m)$ for fermions and from $M(h, m)$ for bosons, and the 'scalar products' have to be replaced by suitable propagators. Details may be found in [10] or any book on quantum field theory dwelling on renormalization.

[19] Or currents, in a generating functional approach; formally, the label $\phi(x)$ should be associated with an element $[\phi|x]$ of a letter-place alphabet in the sense of Rota [22].

It was argued e.g., in [14–16, 18, 21] that the structure of a quantum field theory is governed by two structures: the quantization, a bilinear form of the opposite symmetry type from that of the fields, and the propagator, a bilinear form of the same symmetry type. A detailed account of these findings in quantum field theory in Hopf algebraic terms is given in [9], where also the cohomology is used as *classifying principle*. The same holds true for symmetric function theory, and indeed a broader analogy between the algebraic structures present in the symmetric functions and QFT comes by considering the deforming products. As we defined the branching operators, they may be seen as '*active*' endomorphisms transforming any element of Λ into another such element, especially a Schur function into a series $\{\lambda\}/\Phi$, e.g. $s_{(21)}/M = \sum_{m\in M} s_{(21)/m} = s_{(21)} + s_{(2)} + s_{(1^2)} + s_{(1)}$. Due to our construction, these transformations are invertible. Writing this in a more Hopf algebraic flavour, we had: $\Phi(s_\lambda) = \sum_\lambda \phi(s_{\lambda(1)})s_{\lambda(2)}$, and $\Phi^{-1}(s_\lambda) = \sum_\lambda \phi^{-1}(s_{\lambda(1)})s_{\lambda(2)}$. Looking at the structure of quantum field theory, it was noted [8, 9, 17, 18] that this formula is nothing but a Wick transformation from normal- to time-ordered field operator products. Considering the s_λ as the 'normal-ordered' basis, the above formula computes the 'time-ordered' expression s_λ/M in the 'normal-ordered' basis. Of course, time and normal-ordering is just a name tag in the theory of symmetric functions. However, this opens up a second, *'passive'*, perspective, which we believe to be a new result in the theory of symmetric functions. Define a new *dotted* 'outer product', denoted by $:$ for the moment[20]. The non-augmented outer product algebras $(\Lambda, \cdot)$ and $(\Lambda, :)$ are isomorphic, hence there exists an isomorphism $s_\lambda \cdot \cdots \cdot s_\mu \mapsto s_\lambda : \cdots : s_\mu$. With the artificial terminology, borrowed from quantum field theory, this amounts to saying that the algebra isomorphism transforms the 'normal-ordered' outer product $\cdot$ into the 'time-ordered' dotted outer product $:$. In quantum field theory, however, an additional structure must be taken into account which destroys the isomorphy—namely the unique vacuum described by the counit. The same happens to be true for symmetric functions, where the counit is the evaluation of the symmetric functions at $x_i = 0$ for all x_i. This evaluation is different in time- and normal-ordered expressions, as we demonstrated in (3.63). In summary, the structural analogy with QFT developed is that the 'geometry' –i.e. the analogue of quantization—is induced by the Schur scalar product (or its inverse), while the ordering structure or basis choice is maintained by the 2-coboundaries, or 'propagators' in the QFT language.

This paper has been concerned with symmetric function theory, as a potential laboratory for quantum field theory. Given the importance of symmetry computations in multi-particle quantum systems, as in quark models, the nuclear shell-model, the interacting boson and the vibron model, spectrum generating groups, as well as exactly solvable models in quantum field theory and statistical mechanics, and especially two-dimensional systems and the fermion–boson correspondence, it is perhaps not surprising that a very close analogy can be found. As far as symmetric function theory itself is concerned, questions raised by the present study include for example a deeper Sweedler-cohomological classification of ring extensions, and an associated classification of branching and product rules for the therewith-attached (quantum) (affine) (non-compact) Lie (super) groups? Even the un-bracketed words in this list of attributes come under scrutiny in the light of the above discussion of the possible role of Littlewood's series $\{3\} \circ L$, $\{21\} \circ L$ or $\{1^3\} \circ L$. Applications to extended, possibly non-associative, algebraic structures which may relate to compositeness may be implied by the present framework.

[20] Dotted wedge products were introduced, e.g., in [20, 18]; in the present context the 'dotted' product is of course that developed in section 4; see for example (4.86).

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Appendix. Hopf algebras versus λ-rings

A.1. Definition of λ-ring

Since we used in this paper partly λ-ring notion, it might well serve the reader to have the definitions around and to explore the relations between λ-rings and Hopf algebras a little further. Let **R** be a commutative unital ring. To form a λ-ring the following additional requirements are imposed:

Definition A.1. *A λ-ring is a ring* **R** *supplemented with the following structure maps* $(r, s \in \mathbf{R})$*:*

$$\begin{aligned}
&\lambda^0(r) = 1\\
&\lambda^1(r) = r\\
&\lambda^n(1) = 0 \qquad \forall n > 1\\
&\lambda^n(r+s) = \sum_{p+q=n} \lambda^p(r)\lambda^q(s)\\
&\lambda^n(rs) = \mathsf{P}_n(\lambda^1(r), \ldots, \lambda^n(r); \lambda^1(s), \ldots, \lambda^n(s))\\
&(\lambda^n \circ \lambda^m)(r) = \lambda^n(\lambda^m(r)) = \mathsf{P}_{n\cdot m}(\lambda^1(r), \ldots, \lambda^{n\cdot m}(r))
\end{aligned} \tag{A.88}$$

where P_n *and* $\mathsf{P}_{n\cdot m}$ *are certain universal polynomials with integer coefficients [1, 30].*

In fact, we can give the polynomials P_n and $\mathsf{P}_{n\cdot m}$ in the case of symmetric functions as follows. Let $x = x_1 + x_2 + x_3 + \cdots$ be an *alphabet* of grade 1. Hence, we will consider the λ-ring $\Lambda(x)$ generated by this alphabet x. Introduce a second alphabet $y = y_1 + y_2 + y_3 + \cdots$ and the elementary symmetric functions for these two sets of indeterminates

$$\begin{aligned}
(1 + e_1(x)t + e_2(x)t^2 + \cdots) &= \prod_i (1 + x_i t)\\
(1 + e_1(y)t + e_2(y)t^2 + \cdots) &= \prod_i (1 + y_i t).
\end{aligned} \tag{A.89}$$

Then $\mathsf{P}_n(e_1(x), e_2(x), \ldots e_n(x), e_1(y), e_2(y), \ldots, e_n(y))$ is defined to be the coefficient of t^n in $\prod_{i,j}(1 + x_i y_j t)$. Similarly $\mathsf{P}_{n\cdot m}(e_1(x), e_2(x), \ldots, e_{n\cdot m}(x))$ is the coefficient of t^n in the following product $\prod_{1\leqslant i_1\leqslant i_2\leqslant\cdots\leqslant i_d\leqslant q}(1 + x_{i_1}x_{i_2}\cdots x_{i_d}t)$. Neither polynomial depends on q, r if the number of variables is sufficiently large, and are called universal polynomials.

The well-known relation to formal power series is as follows [24]. Let **R** be any commutative unital ring, then there is a functor Λ : **ring** $\to$ **ring** which assigns to **R** the universal λ-ring $\Lambda(\mathbf{R})$. Consider formal power series of the form

$$f(t) = 1 + \sum_{i\geqslant 1} r_i t^i \qquad g(t) = 1 + \sum_{i\geqslant 1} s_i t^i \tag{A.90}$$

and define addition and multiplication in $\Lambda(\mathbf{R})$ as

$$\begin{aligned} f(t) +_\Lambda g(t) &= 1 + \sum_{i\geqslant 1}\left(\sum_{n+m=i} r_n s_m\right) t^i \\ f(t) \cdot_\Lambda g(t) &= 1 + \sum_{i\geqslant 1} \mathsf{P}_i(r_1,\ldots,r_i;s_1,\ldots,s_i)t^i \end{aligned} \tag{A.91}$$

where P_i is the polynomial appearing in the definition of the λ-ring. Furthermore, for any ring homomorphism $h : \mathbf{R} \to \mathbf{S}$ one defines $H : \Lambda(\mathbf{R}) \to \Lambda(\mathbf{S})$ as

$$H\left(1 + \sum_{i\geqslant 1} r_i t\right) = 1 + \sum_{i\geqslant 1} h(r_i)t^i = 1 + \sum_{i\geqslant 1} s_i t^i \tag{A.92}$$

where the $r_i \in \mathbf{R}$ and the $s_i \in \mathbf{S}$. This turns Λ into an endofunctor on **ring**. The action of λ^i on elements of $\Lambda(\mathbf{R})$ is defined to be

$$\lambda^i(f(t)) = 1 + \sum_{j\geqslant 1} \mathsf{P}_{i\cdot j}(r_1,\ldots,r_{ij})t^j \tag{A.93}$$

where the P_{ij} are the polynomials of the definition of the λ-ring.

Further important λ-ring operations are the Adams operations, defined as

$$\psi^1(r) = r \qquad \psi^n(\psi^m(r)) = \psi^m(\psi^n(r)) = \psi^{n\cdot m}(r). \tag{A.94}$$

Of course, from the definition of the λ-ring and these relations one reads that Adams operations are connected with composition or plethysm and the power sum basis.

The λ-maps can be used to form a ring-map from $\mathbf{R}$ to $\Lambda(\mathbf{R})$ which assigns to every r a formal power series

$$\lambda_t : \mathbf{R} \to \Lambda(\mathbf{R}) \qquad \lambda_t(r) = \sum_{i\geqslant 0} \lambda^i(r)t^i = 1 + \sum_{i\geqslant 1} \lambda^i(r)t^i = 1 + \sum_{i\geqslant 1} r_i t^i. \tag{A.95}$$

With this in mind, it is easily seen how the translation table in Macdonald [35 p 18] occurs. One obtains the translations

$$\begin{aligned} & x = x_1 + x_2 + \cdots + x_n + \cdots \\ & e_r \leftrightarrow \lambda^r(x) \qquad r\text{th exterior power} \\ & E(t) \leftrightarrow \lambda_t(x) \\ & h_r \leftrightarrow \sigma^r(x) \qquad r\text{th symmetric power} \\ & H(t) \leftrightarrow \sigma_t(x) = \lambda_{-t}(-x) \\ & p_r \leftrightarrow \psi^r(x) \qquad \text{Adams operations} \\ & P(t) \leftrightarrow \lambda_{-t}^{-1}(x)\frac{\mathrm{d}}{\mathrm{dt}}\lambda_{-t}(x) = \frac{\mathrm{d}}{\mathrm{dt}}\log\lambda_{-t}(x). \end{aligned} \tag{A.96}$$

Rota and collaborators used letter-place super algebras for their works in invariant theory and combinatorics [22], which is related to the λ-ring formalism as follows. Let L be an alphabet of possibly signed letters—we assume positive letters to avoid sign problems. As can be deduced from [40, 41], the theory of symmetric functions is derived from a letter-place algebra of a single letter x, which therefore has to be considered in terms of λ-ring structures. In fact, the ring Λ of symmetric functions is the free ring underlying the λ-ring $\Lambda(X)$ in a single variable, [35, p 17]. In this sense, Rota's plethystic algebra $\mathsf{Pleth}[L]$ is concerned with those parts of the λ-ring structure which are related with P_{nm}.

A.2. Hopf algebraic aspects of λ-rings

Note that one can assign to a generating function such as $H(t)$ a Toeplitz matrix, that is a band matrix with entries $h_{ij} = [h_{i-j}]$. This morphism from generating functions to matrices turns the pointwise product of ordinary polynomial sequence generating functions (opgf, [49]) into the matrix product of Toeplitz matrices.

$$f(t)\cdot g(t) \cong \begin{bmatrix} f_0 & f_1 & f_2 & f_3 & \cdots \\ 0 & f_0 & f_1 & f_2 & \cdots \\ 0 & 0 & f_0 & f_1 & \cdots \\ 0 & 0 & 0 & f_0 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix} \circ \begin{bmatrix} g_0 & g_1 & g_2 & g_3 & \cdots \\ 0 & g_0 & g_1 & g_2 & \cdots \\ 0 & 0 & g_0 & g_1 & \cdots \\ 0 & 0 & 0 & g_0 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix}$$
$$= \begin{bmatrix} h_0 & h_1 & h_2 & h_3 & \cdots \\ 0 & h_0 & h_1 & h_2 & \cdots \\ 0 & 0 & h_0 & h_1 & \cdots \\ 0 & 0 & 0 & h_0 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix} \tag{A.97}$$

where $h_i = \sum_{n+m=i} f_n g_m$ and $h_i = 0$ if $i < 0$. Now, this can be easily recast in terms of a convolution product of a coalgebra and an algebra map. Let

$$\delta(t) = t \otimes t \qquad \mu(t^r \otimes t^s) = t^{r+s} \tag{A.98}$$

then one finds using maps $F : t \to \sum f_i t^i$ and G analogously,

$$\begin{aligned}(F * G)(t) &= \mu \circ (F \otimes G) \circ \delta(t) = F(t)\cdot G(t) \\ &= \sum_{n+m=i} f_n g_m t^i = \sum_i \sum_r f_r g_{i-r} t^i.\end{aligned} \tag{A.99}$$

Hence the pointwise product of generating functions can be understood as a convolution algebra made from a coalgebra–algebra pair. Note that the convolution *product* is related to λ-ring *addition*. Moreover, one can dualize this approach defining a suitable coproduct acting on the coefficients of the generating functions. Therefore we define

$$\begin{aligned}\Delta(e_n) &= \sum_{r=0}^{n} e_r \otimes e_{n-r} \\ \Delta(e_n) &= \theta^{-1}(\Delta(e_n)(X+Y)) = \sum_{r=0}^{n} \theta^{-1}(e_r(X)e_{n-r}(Y))\end{aligned} \tag{A.100}$$

where we have used the map θ from section 2.4 to make the connection between tensor formulations and formulation in λ-rings.

Aside. The crucial property of the maps λ^i is that they are similar to sequences of binomial polynomials. These binomial sequences have been studied by Rota and collaborators for quite a while [42]. It is shown there that every shift invariant operator gives rise to a set of polynomials in such a way that they are Appell or Scheffer sequences (*loc cit* p 58), hence fulfilling the property

$$p_n(x+y) = \sum_{s+r=n} \binom{n}{r} p_s(x) p_r(y). \tag{A.101}$$

This relation, including the binomial coefficients, is related to exponential generating functions (egf), and hence to the Adams operations and not directly to the λ^i maps. But it can be shown that there are shift operators E^a which act as

$$E^a p_n(x) = p_n(x+a) \tag{A.102}$$

which is exactly the case for the λ_1 and σ_1 series in λ-ring notation

$$F(X)/\lambda_1 = F(X-1) \qquad F(X)/\sigma_1 = F(X+1). \tag{A.103}$$

It would be extremely interesting to have explicitly the details of this relation, which involves umbral calculus and umbral composition. The vertex operator $\Gamma(1) \circ F(X) = F(X)/\sigma_1/\lambda_1$ embodies in combinatorial terms the principle of inclusion and exclusion (PIE), which is understood in Hopf algebraic terms via the theory of *species* developed by Joyal [26].

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